

CHAPTER

6

DISCOUNTING
FUTURE BENEFITS
AND COSTS

Both private and public decisions can have important consequences that extend over time. When consumers buy houses, automobiles, or education, they generally expect to derive benefits and incur costs over a number of years. When the government builds a dam, subsidizes job training, regulates carbon dioxide emissions, or leases the outer continental shelf for oil exploration, it sets in motion impacts that extend over many years. Often analysts have to compare projects with benefits and costs that arise in different time periods. Formally, they have to make intertemporal (across time) comparisons. To do this, analysts *discount* future costs and benefits so that all costs and benefits are in a common metric—the present value. Thus, they can measure and compare the net social benefits of each policy alternative using the *net present value* criterion.

This chapter deals with the practical issues one must know in order to compute the net present value of a project (or policy).¹ It assumes that the *social discount rate*, the rate at which analysts should discount future benefits and costs of a project, is known. As we discuss in Chapter 10, in practice the discount rate is often set for analysts by an oversight agency, such as the Congressional Budget Office in the United States, or the Treasury Board in Canada.

The sections of this chapter cover the following topics: the basics of discounting; compounding and discounting over multiple years; the timing of benefits and costs; long-lived projects and terminal values; comparing projects with different time frames; real versus nominal dollars; relative price changes; and sensitivity analysis in discounting. Appendix 6A presents some shortcut methods for calculating the present value of annuities and perpetuities. Appendix 6B demonstrates how to calculate present values when interest is compounded more frequently than once per period. The topics covered in this chapter are essentially uncontroversial. Readers who are familiar with capital budgeting techniques may want to skip this chapter.

BASICS OF DISCOUNTING**Projects with Lives of One Year**

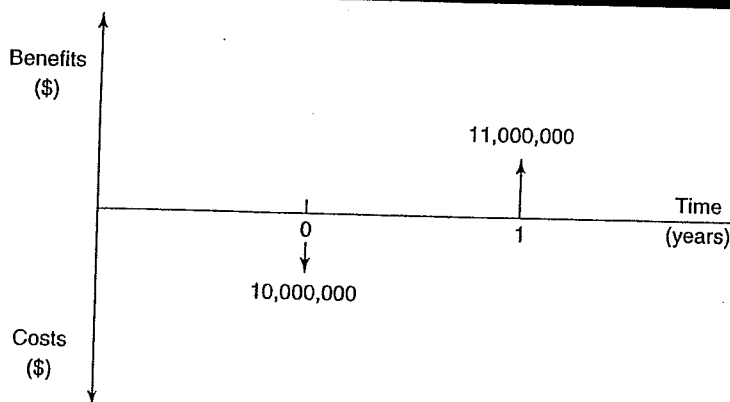
Technically speaking, discounting takes place over *periods* rather than years. But because the period of discounting is a year in almost all applications, and it is easier to think of years rather than periods, we generally use the term *years*. To begin we consider projects that last for exactly one year. In the next section we consider projects that last for longer than one year. In Appendix 6B we discuss how to discount with multiple compounding during a period.

Suppose a state government has the opportunity to buy a parcel of land for \$10 million. It knows that if it buys the land, then the land will be sold for \$11 million one year from now. Should the state buy the land now?

Before proceeding further, it is often useful to lay out the annual benefits and costs of the project on a *time line* as shown in Figure 6.1. A time line is really useful to clarify the timing of the benefits and costs of a project. The horizontal axis represents time measured in years. Benefits appear above the time line, whereas costs are below the time line.

The state should compare the project that has a cost of \$10 million now and a benefit of \$11 million in one year to the most likely alternative—the status quo. There are three ways to do this, each of which gives the same answer.

Future Value Analysis This method compares what the state will receive in the future if it invests in the project with what it will receive in the future if it invests the money in the best alternative. Suppose that if the state does not buy the land it will invest the money in treasury bills (T-bills) at an interest rate of 7 percent.² If it buys the T-bills, it will have \$10.7 million in a year—the principal amount of \$10 million plus interest of $\$10 \text{ million} \times 0.07 = \$700,000$. This amount, \$10.7 million, is called the *future value (FV)* of the T-bills because it represents the amount the state will have in a future period if it buys the T-bills. The state can compare this future value with the future value it will receive if it invests in the land, \$11 million, and choose the alternative that has the highest future value. In this example, the state should buy the land.

FIGURE 6.1 A Time Line Diagram

In general, the future value in one year of some amount X is given by the following formula:

$$FV = X(1 + i) \quad (6.1)$$

where i is the annual rate of interest. The future value is also called the *compound value*. Equation (6.1) illustrates the basic idea of *simple compounding*. We present this method first because the idea of simple compounding is intuitively appealing for anyone who has ever had a savings account. For example, if one invests \$1,000 in a savings account at 5 percent, one will have $\$1,000(1 + 0.05) = \$1,050$ in a year.

Note that interest rates are often stated as percentages, such as 5 percent. This corresponds to an interest rate, i , equal to 0.05.

Present Value Analysis We now switch from compounding to discounting and from future values to present values. Present value analysis compares the amount of money the state must invest today in T-bills in order to have the same amount in a year that it will have if it buys the land with the amount it will invest in T-bills now if it does not buy the land. The *present value (PV)* of the land is the amount the state must invest today in T-bills that yield 7 percent in order to obtain the value of the land in a year, \$11 million. Setting $X = PV$ and $i = 0.07$ in equation (6.1) gives:

$$PV(1 + 0.07) = \$11,000,000$$

Solving this equation for PV gives:

$$PV = \frac{\$11,000,000}{1.07} = \$10,280,374$$

The present value of buying the land is \$10,280,374. In contrast, the present value of the best available alternative, buying the T-bills now, is \$10 million.³ Comparing these two present values shows that the state will be \$280,374 better off in present value terms if it buys the land.

The present value, PV , of an amount that will be received in the future is the current equivalent value of that amount given prevailing interest rates. In general, if the prevailing interest rate is i , then the present value of an amount received in one year is given by:

$$PV = \frac{Y}{1 + i} \quad (6.2)$$

The process of calculating the present value of future amounts is called *discounting*. Comparing equations (6.1) and (6.2) shows that discounting is the opposite of compounding.⁴ As is evident from equation (6.2), the present value of a future amount decreases as the interest rate increases.

Net Present Value Analysis This method calculates the present values of all the benefits and costs of a project, including the initial investment, and sums them to obtain the *net present value (NPV)* of the project. For the land purchase example, the NPV is the difference between the present value of the land and the current cost of the land:

$$NPV = \$10,280,374 - \$10,000,000 = \$280,374$$

These calculations are represented graphically on a time line in Figure 6.2. As the NPV of buying the land is positive, the state should invest in this project. The state will be \$280,374 better off in present value terms if it buys the land.

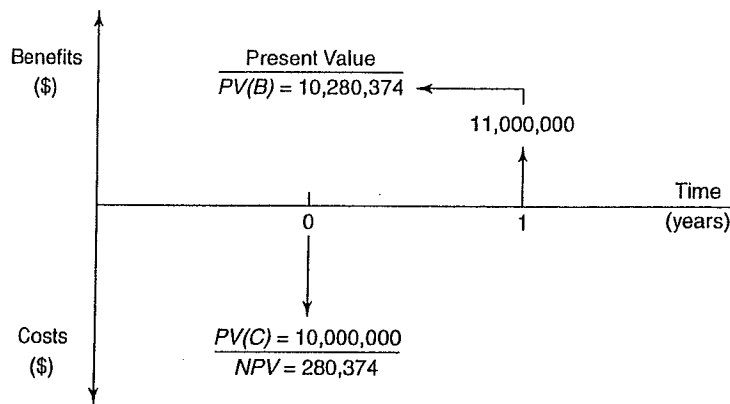


FIGURE 6.2 NPV Calculation

The *NPV* of a project equals the difference between the present value of the benefits, $PV(B)$, and the present value of the costs, $PV(C)$:

$$NPV = PV(B) - PV(C) \quad (6.3)$$

As mentioned in Chapter 1, the *NPV* method provides a simple criterion for deciding whether to undertake a project. If the *NPV* of a project is positive, then one should proceed with it; if the *NPV* is negative, then one should not. The positive *NPV* decision rule assumes implicitly that there is no other alternative with a higher *NPV*. *If there are multiple, mutually exclusive alternatives, then one should select the alternative with the highest NPV.*

The foregoing example assumes that the state has \$10 million available that could be used either to buy the land or to invest at interest rate i . Sometimes analysts calculate *NPVs* of projects for which the government may not have all the cash immediately available and it may have to borrow some funds. *Implicitly, analysts assume that the government can borrow or lend funds at the same interest rate i . Under this assumption it does not matter whether the government currently has the money or not: The *NPV* rule still holds.* In Chapter 10 we discuss how the source of funding for a project may affect the choice of the discount rate. However, even in these situations, analysts should select the project with the largest *NPV*.

COMPOUNDING AND DISCOUNTING OVER MULTIPLE YEARS

We now generalize these results across many years. As before we first discuss future values, then present values, and finally net present values.

Future Value over Multiple Years Suppose that the state could invest the \$10 million for five years with interest at 7 percent per annum compounded annually. Using equation (6.1), at the end of the first year the state would have \$10 million \times 1.07 = \$10.7 million. Again using equation (6.1), at the end of the second year the state would have \$10.7 million \times 1.07 = \$11.449 million. Notice that the interest in the second year, \$0.749 million, is more than the interest in the first year, \$0.700 million. In the

to yield \$100,000 in three years, denoted by PV , can be found by substituting into equation (6.4):

$$PV(1 + 0.06)^3 = \$100,000$$

Solving this equation for PV gives:

$$PV = \frac{\$100,000}{(1 + 0.06)^3} = \frac{\$100,000}{1.19102} = \$83,962$$

Consequently, the government branch would need \$83,962 now to have \$100,000 in three years.

In general, the present value of an amount received in n years, denoted Y , with interest compounded annually at rate i is:

$$PV = \frac{Y}{(1 + i)^n} \quad (6.5)$$

The term $1/(1 + i)^n$, which equals the present value of \$1 received in n years when the interest rate is i , is called the *present value factor* or the *discount factor*. For example, the present value factor in the foregoing example equals $1/(1 + 0.06)^3 = .8396$. Again, these factors are available in most finance textbooks, on handheld calculators, and in computer spreadsheet programs.

If a project yields benefits in many periods, then we can compute the present value of the whole stream by adding the present values of the benefits received in each period. Specifically, if B_t denotes the benefits received in period t for $t = 0, 1, \dots, n$, then the present value of a stream of benefits, denoted $PV(B)$, is:

$$PV(B) = \frac{B_0}{(1 + i)^0} + \frac{B_1}{(1 + i)^1} + \dots + \frac{B_{n-1}}{(1 + i)^{n-1}} + \frac{B_n}{(1 + i)^n}$$

$$PV(B) = \sum_{t=0}^n \frac{B_t}{(1 + i)^t} \quad (6.6)$$

Similarly, if C_t denotes the costs incurred in period t for $t = 0, 1, \dots, n$, then the present value of a stream of costs, denoted $PV(C)$, is:

$$PV(C) = \sum_{t=0}^n \frac{C_t}{(1 + i)^t} \quad (6.7)$$

To illustrate the use of equation (6.6), consider a government agency that has to choose between two alternative projects. Project I yields a benefit of \$10,500 four years from now whereas project II yields \$5,500 four years from now and an additional \$5,400 five years from now. Assume the interest rate is 8 percent. Which is the better project? The present values of each project are:

$$PV(I) = \frac{\$10,500}{(1 + 0.08)^4} = \$7,718$$

$$PV(II) = \frac{\$5,000}{(1 + 0.08)^4} + \frac{\$5,400}{(1 + 0.08)^5} = \$4,043 + \$3,675 = \$7,718$$

In this example, the present values of the two projects happen to be identical. Time lines for these projects are shown in Figure 6.3.

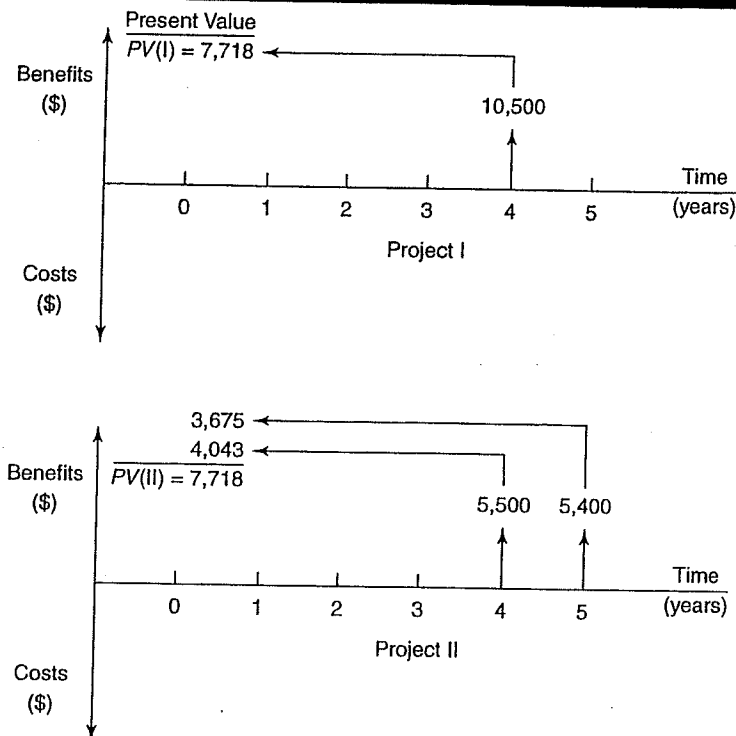
Net Present Value of a Project We have now introduced all of the material on basic discounting needed for CBA. As discussed earlier, the *NPV* of a project is the difference between the present value of the benefits and the present value of the costs, as represented in equation (6.3). Substituting equations (6.6) and (6.7) into equation (6.3) gives the following useful expression:

$$NPV = \sum_{t=0}^n \frac{B_t}{(1+i)^t} - \sum_{t=0}^n \frac{C_t}{(1+i)^t} \quad (6.8)$$

To illustrate the mechanics of computing the *NPV* of a project using this formula, suppose a district library is considering purchasing a new information system that will give users access to a number of online databases for five years. The benefits of this system are estimated to be \$100,000 per annum, including both cost savings to the library and user benefits. The information system costs \$325,000 to purchase and set up initially and \$20,000 to operate and maintain each year. After five years, the system will be dismantled and sold for \$20,000.⁷ Assume the appropriate discount rate is 7 percent and there are no other costs or benefits.

A time line for this project is shown in Figure 6.4. It shows the timing of each of the benefits and costs, their present values, the present value of all the benefits, the pre-

FIGURE 6.3 Comparing Project I and Project II



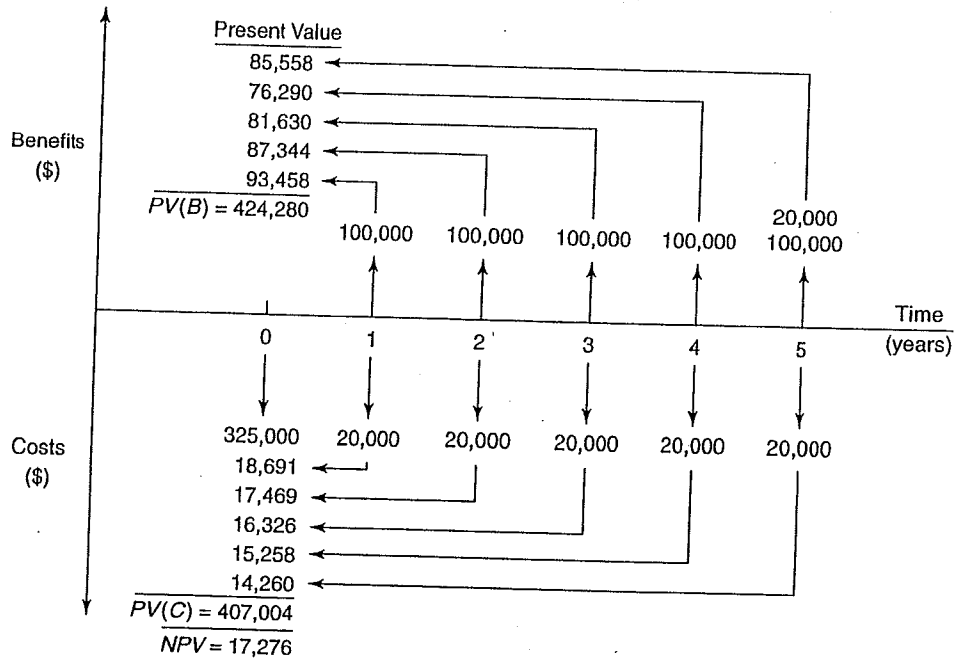


FIGURE 6.4 Time Line of the Benefits and Costs of the Library Information System

present value of all the costs, and the *NPV* of the project. The present value of the benefits is \$424,280, the present value of the costs is \$407,004, and the *NPV* of the project is \$17,276. As the *NPV* is positive, the library should purchase the new information system.

An alternative way to compute the *NPV* of a project is to compute the present value of the annual *net benefits* (*NB*). Let $NB_t = B_t - C_t$ denote the annual net benefits arising in year t ($t = 0, 1, 2, \dots, n$). It follows from equation (6.8) that the *NPV* of a project equals the present value of the net benefits:⁸

$$NPV = \sum_{t=0}^n \frac{NB_t}{(1+i)^t} \tag{6.9}$$

To illustrate that equation (6.9) and equation (6.8) produce the same *NPV*, Table 6.2 contains the annual benefits, annual costs, and annual net benefits of the library information system project. Using equation (6.9), the present value of the net benefits of the project is \$17,276, as shown in the last column of Table 6.2. This is the same value as the *NPV* obtained earlier by taking the difference between the present value of the benefits and the present value of the costs.

In many respects, tables and time lines are interchangeable. For comparison purposes a time line using net benefits appears in Figure 6.5. Tables and time lines present key information succinctly and facilitate computation of project *NPVs*. Neither is necessary. Analysts can experiment with them and use them whenever they are helpful. An advantage of time lines is that they indicate precisely when impacts occur during a year.

TABLE 6.2 The Net Present Value of the Library Information System

Year	Event	Annual Benefits	Annual Costs	Annual Net Benefits
0	Purchase and install	0	325,000	-325,000
1	Annual benefits and costs	100,000	20,000	80,000
2	Annual benefits and costs	100,000	20,000	80,000
3	Annual benefits and costs	100,000	20,000	80,000
4	Annual benefits and costs	100,000	20,000	80,000
5	Annual benefits and costs	100,000	20,000	80,000
5	Liquidation	20,000	0	20,000
<i>PV</i>		\$424,280	\$407,004	\$ 17,276

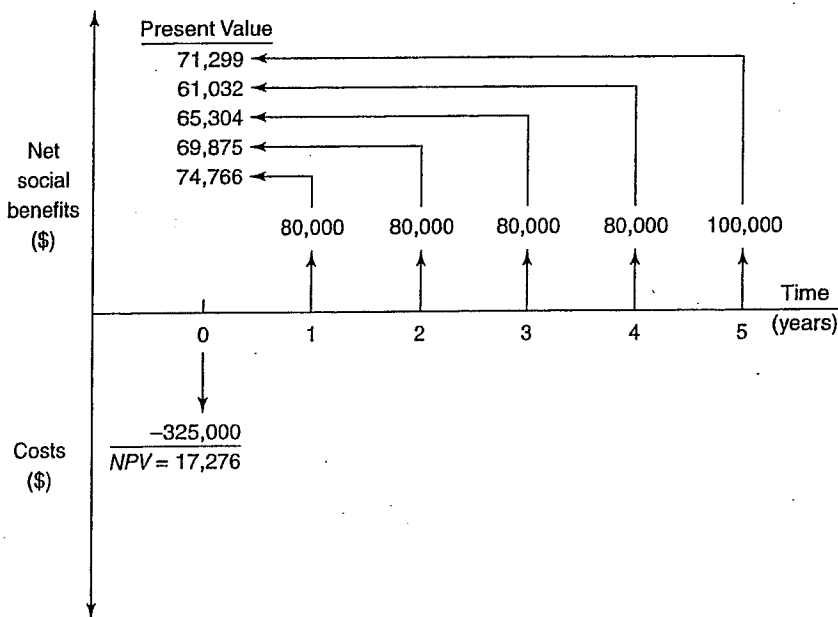


FIGURE 6.5 Time Line of the Net Social Benefits of the Library Information System

Two special situations are worth discussing briefly. For some projects all of the costs occur immediately ($t = 0$) and only benefits occur in the ensuing years ($t = 1, 2, \dots, n$). In this situation, equation (6.8) simplifies to:

$$NPV = \sum_{t=0}^n \frac{B_t}{(1+i)^t} - C_0$$

For a project with some impacts that last indefinitely, we replace n in equation (6.8) or (6.9) with infinity, ∞ . Alternatively, we discount over some finite period and use a ter-

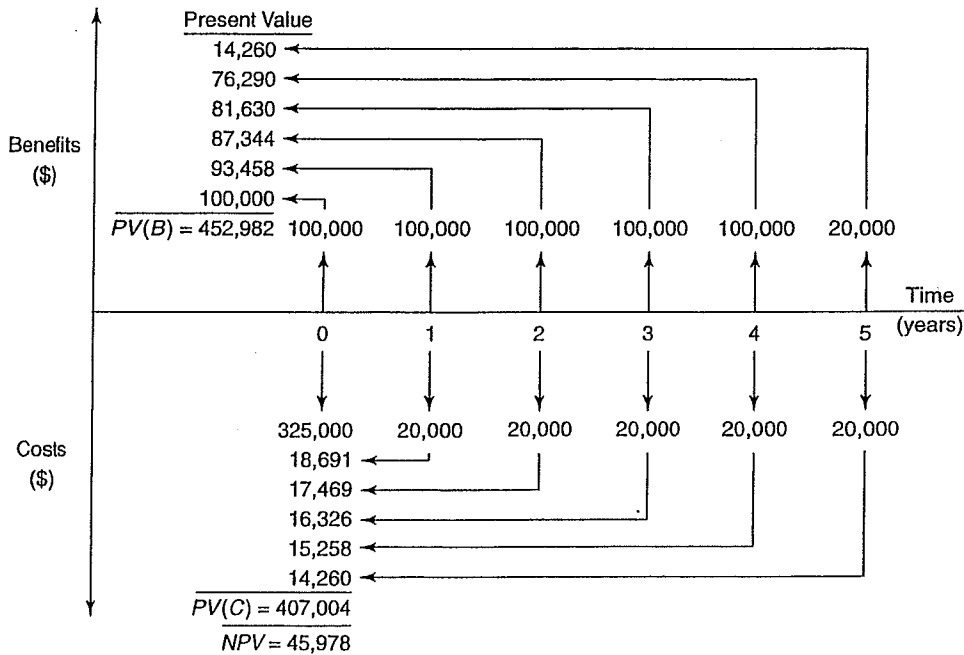
minimal value to capture the present value of subsequent net benefits. Estimation of terminal values is discussed later in this chapter.

TIMING OF BENEFITS AND COSTS

The compounding and discounting formulas presented so far assume that all benefits and costs occur at the end of each period (year). Impacts are assumed to arise immediately ($t = 0$), or at the end of the first year ($t = 1$), or at the end of the second year ($t = 2$), and so on. For many projects this is a reasonable assumption. Furthermore, when most of the costs occur early in the project and most of the benefits occur late in the project, this assumption is conservative in the sense that the *NPVs* are lower than they would be if they were computed under alternative assumptions.

To illustrate this point, reconsider the library information system example, but now assume that the annual benefits of \$100,000 all occur at the beginning of each year instead of at the end of each year, and assume the timing of all other benefits and costs is unchanged. A time line under these assumptions is given in Figure 6.6. The present value of the benefits has increased by \$28,702 from \$424,280 to \$452,982, and the *NPV* of the project has increased by \$28,702 from \$17,276 to \$45,978, more than two and a half times the initial amount. Clearly, the *NPV* can vary considerably according to the assumption made about the timing of benefits or costs.

FIGURE 6.6 Time Line of Benefits and Costs of the Library Information System Assuming User Benefits Occur at the Beginning of Each Year



A more reasonable assumption might be to assume that the benefits occur throughout the year. The average of the two *NPVs*, $(\$17,276 + \$45,978)/2 = \$31,627$, provides an estimate of the *NPV* under this assumption.

LONG-LIVED PROJECTS AND TERMINAL VALUES

Earlier we stated that analysts should discount benefits and costs over “the life of the project” using equation (6.8) or, equivalently, equation (6.9). These equations imply that all of the impacts attributable to the project have occurred during the first n years—the life of the project. Subsequent benefits and costs are assumed to equal zero.

Even though a project may be finished from an engineering or administrative perspective after a relatively short period of time, the benefits (and some costs) may continue to flow from the project for many years. In England, for example, cars travel on roads that were laid out by the Romans—more than 15 centuries ago. The Great Wall of China continues to generate tourism benefits even though it was built many centuries ago. The same issue also arises in human capital investment programs, especially training and health programs. For example, preschool training programs may benefit participants throughout their entire lives, years after they participated in the program; some benefits may even accrue to their children. All of these impacts should be included in a CBA. In practice, it is not clear how to handle costs and benefits that arise far in the future.

One option is to estimate the benefits and costs (or the net benefits) in each and every period and to calculate the *NPV* using the following formula:

$$NPV = \sum_{t=0}^{\infty} \frac{NB_t}{(1+i)^t} \quad (6.10)$$

It is practical to use this method if it is reasonable to assume that the net benefits are constant or grow at a constant rate; see Appendix 6A. But these assumptions may not be appropriate. Furthermore, as analysts are likely to be more confident about predicting the “near future” than the “far future,” it is often useful to distinguish between them.

For most long-lived projects, analysts prefer to select a *relatively* short discounting period (often the *useful life* of the project), include a *terminal value* to reflect all subsequent benefits and costs, and perform *sensitivity analysis* on the terminal value. Thus, analysts distinguish between “near future” impacts and “far future” impacts. The latter are reduced to a single number, the terminal value. Formally, if k denotes the number of discounting periods and $T(k)$ denotes the terminal value, then we use the following formula to compute the *NPV*:

$$NPV = \sum_{t=0}^k \frac{NB_t}{(1+i)^t} + T(k) \quad (6.11)$$

The terminal value is the net present value of all the benefits and costs that occur after the discounting period.⁹ Because of uncertainty concerning the actual magnitude of the terminal value, sensitivity analysis is often conducted by selecting alternative terminal values and seeing how this affects the findings.¹⁰

When using terminal values, the length of the discounting period, k , is arbitrary in theory.¹¹ In practice, the discounting period is usually determined by the nature of each project. For example, it is common to use a 20-year discount period for highways because they tend to last about 20 years before they require major repairs.

Alternative Methods for Estimating Terminal Values

There is a variety of methods for estimating terminal values. One method is based on simple projections, another uses scrap values or liquidation values, a third uses (economic) depreciated values, a fourth is based on initial construction costs, and a fifth assumes the terminal value is zero.

Terminal Values Based on Simple Projections One theoretically appropriate method estimates the terminal value based on simple extrapolations of benefits and costs (or net benefits). This is similar to estimating equation (6.10) directly, but it distinguishes between the "near future" and the "far future." Consider, for example, the construction of a new dike and suppose the annual net benefits have been calculated for the first 35 years. It is necessary to estimate a terminal value for the period after 35 years. One possibility is to make an assumption about the growth in future net benefits. Suppose analysts expect net benefits will be \$1 million in the 36th year and they will grow at 1.5 percent per annum indefinitely. Using equation (6A.7) in Appendix 6A, the formula for the PV of a perpetuity that grows at a constant rate, and an interest rate of 8 percent, the value at the end of the 35th year of the subsequent net benefits is $\$1 / (0.008 - 0.015)$ million = \$15.38 million. The present value of these net benefits is \$1.04 million.

For many government projects, especially training programs, it is reasonable to assume that the annual net benefits decay at a constant rate after some date. In this case equation (6A.7) can be used with a negative growth rate.

Terminal values obtained by this method are usually very sensitive to the discount rate and the growth rate, again emphasizing the importance of sensitivity analysis. However, there is strong evidence that, for medium- to long-term periods, simple forecasting models predict better than more complicated models. Thus, this method may estimate the NPV at least as well as more complicated direct estimation of equation (6.10).

Terminal Values Based on Salvage Value or Liquidation Value For some projects, the scrap value, residual value, liquidation value, or salvage value of the plant and equipment may be used as the terminal value. For example, a school board may buy some buses that will last 25 years at which time they will be sold at market value for use by someone else or for scrap. This liquidation value or scrap value may be used as the terminal value. This method is appropriate when no other (social) costs or benefits arise beyond the discounting period, when there is a well-functioning market in which to value the asset, and when the market value reflects the asset's social value (e.g., no negative externalities).

Notice that many short projects have terminal values. For example, the library information system would be sold for \$20,000 after only five years. This amount is a terminal value. Analysts should consider whether \$20,000 is an accurate measure of the net social benefits of the equipment at liquidation. For example, if the equipment would be used in schools where its social value (less new set-up costs) would be more than \$20,000, then a higher terminal value should be used.

In practice, it may be very difficult to determine the liquidation value of an asset. Consider estimating a terminal value for a highway project at the end of its useful life of, say, 20 years. Clearly there is no market for used highways and, even if there was one, it would probably not reflect the discounted value of future net social benefits.

Estimating Terminal Values Based on Depreciated Value The third method recognizes that the stream of benefits and costs from a capital-oriented project is directly related to its depreciated value. Indeed, by definition, the value of an asset equals the present value of the net benefits that it generates. However, rather than focus on estimating the stream of benefits and costs, this method focuses on estimating the depreciated value.¹²

It is important to emphasize that we are referring to real (i.e., economic) depreciation, not accounting depreciation. There may be a huge difference. Economic depreciation concerns the decline in the economic value of an asset over time. In contrast, accounting depreciation is largely determined by tax or reporting requirements. Tax authorities may allow companies to take 100 percent depreciation in one year, in which case a company can write off 100 percent of the cost of an investment, even though the investment itself may yield benefits that extend over decades. Thus, the depreciated accounting value may bear no relationship to the reduced usefulness or the amount of wear and tear of an asset. *Accounting depreciation should never be included as a cost (expense) in CBA.*

Using economic depreciation value is applicable where there is no market for some capital item so that it remains in the public sector. However, this method suffers from three problems:

1. Estimating economic depreciation rates can be very difficult. Different assets depreciate at different rates. For example, aircraft are maintained at near 100 percent efficiency until they fall apart; in contrast, the efficiency of railroads initially declines very quickly and then decreases at a decreasing rate.
2. The amount of economic depreciation is often endogenous: Depreciation is affected by how the asset is used and maintained in the project. If usage is low and maintenance is high, then an asset may continue to perform at 100 percent efficiency for many years. If usage is high and maintenance is low, then it may start to fall apart quickly. Thus, a capital asset's value depends on the project itself.
3. The rate at which a piece of equipment or a project actually declines may bear no relationship to the stream of social benefits and costs that it generates. For example, an aircraft may be maintained at 100 percent efficiency but, if it is in "mothballs" and nobody flies in it, the social benefits may be zero.

Estimating Terminal Values Based on the Initial Construction Cost This method estimates the terminal value based on initial construction costs. In the highway example we present in Chapter 1, we discount the annual benefits over 20 years, the expected useful life of the project before major repairs. The terminal value of the highway *at the end of the discounting period* was assumed to equal 75 percent of initial construction costs: $0.75 \times \$338.1 \text{ million} = \253.58 million . That is, the future value of the highway in 21.5 years (20 years after construction *was completed*) was assumed to equal \$253.58 million, which has a present value in 1986 dollars of \$53.3 million. Consequently, ac-

According to the method basing terminal values on initial construction cost, the terminal value is \$53.3 million.

In effect, this method is a special case of using depreciated values. When using this method, the analyst must select some proportion of the initial construction costs to use as a terminal value. However, there is no evidence to suggest that the present value of the net social benefits of a highway after 20 years is related to initial construction costs at all. The 75 percent figure is quite arbitrary. This method then is not intuitively appealing.

Setting the Terminal Value Equal to Zero A final method chooses a fairly long discounting period and ignores subsequent benefits and costs. In effect, this is a special case of the first method we discussed—it is equivalent to assuming that after k periods the net benefits in each subsequent period are zero. In private-sector decision making, this may be a reasonable assumption because project evaluation requires only the consideration of private benefits and costs that may approach zero fairly quickly. But the social impacts of government projects may last many years. Analysts may omit important benefits or costs if they use a time horizon that is too short.

Conclusion Concerning Terminal Values

The analyst must decide on both the discounting period and the method for calculating the terminal value. *We suggest analysts discount over the useful life of the project and estimate the terminal value on the basis of simple projections that rely on reasonable assumptions.* The other methods that are used for computing terminal values are more difficult in practice or are more ad hoc. Thus, in practice, the estimated terminal value may bear little relation to the theoretically correct amount. *One should usually perform sensitivity analysis on the terminal value.*

The length of the discounting period and the method for calculating the terminal value may be interdependent. If, for example, the analyst is going to assume the terminal value is zero, then he or she should use a relatively long discounting period. If the analyst is using one of the other methods, then it makes sense to discount over the project's useful life. For physical projects this information is usually provided by engineers.

COMPARING PROJECTS WITH DIFFERENT TIME FRAMES

Projects with different time frames are not directly comparable. They should always be compared over the same discounting period.

Lack of Comparability between Projects with Different Time Frames

Suppose an electric utility company is considering two alternative proposals for new sources of energy. One is a major hydroelectric dam (*HE*), which would last 75 years; the other is a cogeneration plant, which would last 15 years. After considering all relevant social benefits and costs, and assuming a discount rate of 8 percent, the *NPV* of the 75-year hydroelectric project is \$30 million and the *NPV* of the 15-year cogeneration project is \$24 million. Is the hydroelectric project preferable simply because it has the larger *NPV*? The answer is "no." These projects are not commensurable because

they have different life spans. The smaller project could be “rolled over” five times within the life of the hydro project.

There are two methods for evaluating projects with different time frames: rolling over the shorter project and the *equivalent annual net benefit method*. They always lead to the same conclusion, as we now illustrate.

Rolling Over the Shorter Project

Suppose that the utility decides to build the cogeneration power plant. Further suppose that in 15 years it builds a new cogeneration plant, in 30 years it builds another new cogeneration plant, and again in years 45 and 60. If so, the length of these five sequential cogeneration plants will be the same as the length of the hydroelectric project. This makes the projects comparable.¹³

The *NPV* of five back-to-back cogeneration power plants is:

$$\begin{aligned} NPV(5CP) &= \$24 + \frac{\$24}{(1 + 0.08)^{15}} + \frac{\$24}{(1 + 0.08)^{30}} + \frac{\$24}{(1 + 0.08)^{45}} + \frac{\$24}{(1 + 0.08)^{60}} \\ &= \$34.94 \text{ million}^{14} \end{aligned}$$

As this *NPV* is higher than the *NPV* of the hydroelectric project, the utility should build the cogeneration plant.

Equivalent Annual Net Benefit Method

An often easier way to compare projects of unequal lengths is the *equivalent annual net benefit (EANB) method*. The *EANB* of an alternative equals its *NPV* divided by the *annuity factor* that has the same life as the project (i.e., the present value of an annuity of \$1 per year for the life of the project, discounted at the rate used to calculate the *NPV*):

$$EANB = \frac{NPV}{a_i^n} \quad (6.12)$$

where a_i^n is the annuity factor, which is defined by equation (6A.2). The *EANB* is the amount which, if received each year for the life of the project, would have the same *NPV* as the project. For example, the *EANBs* for the hydroelectric (*HE*) and the cogeneration (*CG*) projects equal:¹⁵

$$EANB(HE) = \$30/12.4611 = \$2.407$$

$$EANB(CG) = \$24/8.559 = \$2.804$$

The *EANB* of the cogeneration project is \$2.804 million, which implies that this project is equivalent to an annuity of \$2.804 million per year for 15 years. In contrast, the net benefit of the hydroelectric alternative is equivalent to an annuity of \$2.407 million per year for 75 years. If one could continuously replace each project at the end of its life with a similar project, the cogeneration project would yield net annual benefits equivalent to \$2.804 million per year indefinitely and the hydroelectric project would yield annual net benefits equivalent to \$2.407 million per year indefinitely. Consequently, the cogeneration alternative is preferable, assuming continuous replacement is possible.

An Additional Advantage of the Cogeneration Project

In fact, if the utility chooses the cogeneration project, it may not be desirable to replace it with an identical cogeneration plant in 15 years. At that time a more efficient alternative may be available. In contrast, if the utility builds the hydroelectric project, then it is locked in for 75 years. Thus, the cogeneration project has an additional benefit because of its flexibility in allowing the introduction of more efficient technology if it becomes available during the 75-year period. Chapter 7 discusses such benefits, called quasi-option value, in more depth. Here, it is sufficient to recognize that this alternative has an additional benefit that is not incorporated in the *EANB*.

REAL VERSUS NOMINAL DOLLARS

Conventional private-sector financial analysis measures revenues, expenditures, net income, assets, liabilities, and cash flows in terms of historical monetary units. Such units are referred to as *nominal dollars* (sometimes called *current dollars*). However, if you have ever listened to an older person reminisce, then you probably know that a dollar purchased more in 1970 than it does now—"a dollar's not worth a dollar anymore!" For example, nominal per capita disposable personal income in the United States was four-and-one-half times higher in 1990 than in 1970 (\$3,521 versus \$16,236), but you could not buy four-and-one-half times as many goods and services on an average income in 1990 as you could on an average income in 1970. The purchasing power of a dollar declines with price inflation. In order to control for the declining purchasing power of a dollar due to inflation, we convert nominal dollars to *real dollars* (sometimes called *constant dollars*).

To obtain real, or constant, dollar measures, analysts adjust for changes in inflation. In effect, analysts *deflate* dollars to account for higher prices. The consumer price index (CPI) is the most commonly used *deflator*.¹⁶ It is expressed as the ratio of the cost of purchasing a standard market basket of goods in a particular year to the cost of purchasing the same (or very similar) basket of goods in some base year, multiplied by 100. The CPI for the United States since 1970 is shown in Table 6.3.¹⁷ Currently, the base year in the United States is 1982–84; thus, for the period 1982–84 the *CPI* = 100. Because the CPI for 1970 was 38.8, this implies that the cost of a basket of goods in 1970 was only 38.8 percent of the price of a similar basket of goods in 1982–84. Similarly, because the CPI for 1990 was 130.7, this implies that the price of a basket of goods in 1990 was 130.7 percent of the price in 1982–84 for a similar basket. Thus, the price of a basket of goods was $130.7/38.8 = 3.37$ times higher in 1990 than in 1970. Consequently, returning to our example, a person on a salary of \$3,521 in 1970 would be able to purchase in 1990 the same (or similar) basket of goods as a person on a salary of $\$3,521 \times 3.37 = \$11,865$. Because the average income was \$16,236 in 1990, we can conclude that effective (i.e., real) incomes increased by $(\$16,236 - \$11,865)/\$11,865 = 0.37$, or 37 percent from 1970 to 1990.

In order to convert amounts measured in nominal dollars for some year into amounts measured in real dollars for the base year (1982–84), we simply divide by the CPI for that year (divided by 100). For example, the average real income of people in 1970 measured in 1982–84 dollars was $\$3,521/0.388 = \$9,074$, and the average real income of people in 1990 measured in 1982–84 dollars was $\$16,236/1.307 = \$12,422$.

TABLE 6.3 The U.S. Consumer Price Index (CPI)
All items
1982-84 = 100

Year	CPI	% Change	Year	CPI	% Change	Year	CPI	% Change
1970	38.8	5.7	1980	82.4	13.5	1990	130.7	5.4
1971	40.5	4.4	1981	90.9	10.3	1991	136.2	4.2
1972	41.8	3.2	1982	96.5	6.2	1992	140.3	3.0
1973	44.4	6.2	1983	99.6	3.2	1993	144.5	3.0
1974	49.3	11.0	1984	103.9	4.3	1994	148.2	2.6
1975	53.8	9.1	1985	107.6	3.6	1995	152.4	2.8
1976	56.9	5.8	1986	109.6	1.9	1996	156.9	3.0
1977	60.6	6.5	1987	113.6	3.6	1997	160.5	2.3
1978	65.2	7.6	1988	118.3	4.1	1998	163.0	1.6
1979	72.6	11.3	1989	124.0	4.8	1999	166.6	2.2

Source: <ftp://ftp.bls.gov/pub/special.requests/cpi/cpi.ai.txt>

To convert these amounts from base-year dollars to, say, 1994 dollars, they are multiplied by the CPI for 1994, which is 148.2 (divided by 100):

$$(\$3,521/0.388) \times 1.482 = \$13,449$$

$$(\$16,236/1.307) \times 1.482 = \$18,410$$

Thus, the average real incomes of people in 1970 and 1990, expressed in 1994 dollars, were \$13,449 and \$18,410, respectively. More generally, as the preceding example illustrates, to convert amounts expressed in year *a* nominal dollars into amounts expressed in year *b* real dollars, the year *a* dollar amounts are divided by the CPI for year *a* and multiplied by the CPI for year *b*.

Past Problems with the CPI

Although the CPI is the most widely used measure of inflation, it has historically overstated the rate of increase in the cost of living. Recently, a commission set up by the Senate Finance Committee and chaired by Michael Boskin estimated that in the United States the CPI overestimated inflation by about one percentage point per annum, with a range between 0.8 percentage point and 1.6 percentage points.¹⁸ As a result, people receiving entitlements with increases linked to the CPI were, in effect, receiving more than was originally intended.

There were many reasons for the upward bias in the CPI in the past.¹⁹ One problem was that the CPI did not accurately reflect consumers' purchases. In the pharmaceutical area, for example, when patents expire some consumers switch to generic drugs, which are often as effective as patented drugs but considerably less expensive. As this switch was not immediately picked up by the CPI, the cost of living was overestimated.²⁰ A second problem was that the CPI did not reflect changes in product quality or improvements in the quality of living due to new products.

Economists and statisticians are correcting these problems. In 1998 the U.S. CPI underwent some major revisions.²¹ Nonetheless, we suggest that, when using historical

CPI data, analysts should convert amounts measured in nominal dollars to amounts measured in real dollars in two ways: using the official CPI as suggested earlier in this section and using a price deflator that increases at one percentage point per annum less than the CPI. For example, someone with an income of \$100,000 in 1994 would have a real income in 1999 dollars of \$112,416 using the official CPI, but an income of \$107,018 using a price deflator that increases at one percentage point per annum less than the CPI.²²

Analyzing Future Benefits and Costs in CBA

Analysts may project benefits and costs in either real dollars or nominal dollars. Also, they may use a real interest rate or a nominal interest rate. Care must be taken to ensure that the units of measurement of benefits and costs are consistent with the units of measurement of the discount rate. *If benefits and costs are measured in nominal dollars, then the analyst should use a nominal discount rate; if benefits and costs are measured in real dollars, then the analyst should use a real discount rate.* Both methods result in the same numerical answer.²³

In the private sector, it is more natural to work in nominal dollars. Interest rates and other market data are expressed in nominal dollars, *pro forma* income and cash flow projections make more sense in nominal dollars, and the tax system is based on nominal amounts. However, for analysis of public policy projects, *it is usually easier and more intuitively appealing to express all benefits and costs in real dollars and to discount using a real discount rate.* Returning to our library example, it makes more sense to think about the current and future annual benefits in terms of cost savings to the library at today's prices and to think about user benefits in terms of the number of hours of use at today's value per hour than it does to think about these benefits in terms of future prices or values. If one expects user benefits will actually increase over time, for example, due to more people using the system or because each person uses it more often as he or she learns its capabilities, then the projected real annual benefits will increase. This would be clear if annual benefits and costs were in real dollars. If, alternatively, the annual benefits were expressed in nominal dollars, and they were increasing, it might not be immediately obvious whether the increases were due to increases in real benefits or were due to inflation.

If the analyst prefers to work in real dollars, but benefits, costs, or the interest rate are expressed in nominal dollars, then the amounts in nominal dollars must be converted to real dollars. To convert future benefits and costs from nominal dollars to real dollars, we use the formula for computing present values, equation (6.5), but discount at rate m , where m is the expected rate of inflation during the project.²⁴ To convert a nominal interest rate, i , to a real interest rate, r , with an expected inflation rate, m , we use the following equation:²⁵

$$r = \frac{i - m}{1 + m} \quad (6.13)$$

For example, if the nominal interest rate is 10 percent and inflationary expectations are 4 percent, then the real interest rate is $[(0.10 - 0.04)/1.04] \times 100 = 5.77$ percent.

If expected inflation is quite low (m is small), the real interest rate *approximately* equals the nominal interest rate minus the expected rate of inflation: $r \approx i - m$. For

example, if the nominal interest rate is 10 percent and inflationary expectations are 4 percent, then the real interest rate is approximately 6 percent.

In order to convert benefits or costs from real dollars to nominal dollars, analysts can use the formula for computing future values, equation (6.4), but compound at the expected rate of inflation. To convert a real interest rate to a nominal interest rate, solve equation (6.13) for i .

Estimates of Expected Inflation

Moving from real interest rates or dollars to nominal interest rates or dollars, or vice versa, requires an estimate of the *expected* rate of inflation *during the life of the proposed project*. Analysts often use the current CPI as an estimate of future inflation. However, this estimate will be too low when inflation is increasing and too high when inflation is decreasing. It would be better to use one of a number of widely available forecasts, although these are often available only for one-year or two-year forecasts.²⁶

Inflation forecasts are available from reputable investment firms, branches of the federal government, a Federal Reserve Bank, or the OECD (Organization for Economic Cooperation and Development). Each week *The Economist* presents recent changes in consumer prices and the results of a poll of consumer price forecasts for the current year and the following year.

In the United States there are three easily accessible survey measures of inflationary expectations: the Livingston survey of professional economists, the Michigan survey of households, and the Survey of Professional Forecasters (SPF).²⁷ In a recent article, Lloyd Thomas compares the one-year predictive performance of these surveys and two simple naive alternatives, one using the latest 12-month CPI figure and the other based on the Fisher model of interest rates, equation (6.13).²⁸ He finds that the surveys perform better than the naive models, with the *median* household forecasts performing best. He notes that the forecasts of professional economists have improved over time, and he suggests that the SPF is likely to become increasingly popular. Of particular interest, SPF respondents have been asked to provide 10-year-ahead inflation forecasts of the CPI since 1991.

A Practical Example: Garbage Trucks

A practical example illustrates the basic issues in moving from market interest rates, which are nominal rates, to real interest rates. Consider a city that uses a rural landfill to dispose of solid refuse. By adding larger trucks to the refuse fleet, the city would save \$100,000 in disposal costs during the first year, and *equivalent amounts* in each successive year. The trucks would be purchased today for \$500,000 and would be sold after four years when the city will open a resource recovery plant that will obviate the need for landfill disposal. The real liquidation value of the trucks (i.e., the current market value of four-year-old trucks of the same type and quality as the city would buy) is \$200,000. The city can currently borrow money at a market interest rate of 10 percent. Analysts generally expect that inflation will be 4 percent during the next four years. Should the city buy the trucks? As usual, the answer should be "yes" if the *NPV* is positive. Is it?

Using Real Dollars

The annual benefits and costs in real dollars are given in column 3 of Table 6.4. It is assumed that the annual savings are the same in real terms each year. This assumption is based on several implicit assumptions, for example, the amount of vehicle operating maintenance does not change over the period, and, if the city does not buy the larger trucks, it will not have to pay for overtime. Furthermore, it implicitly assumes that the *relative valuations* (prices) of wages, gasoline, and other components that figure into the benefit calculations do not change over time.

As benefits and costs are expressed in real dollars, we need to use a real discount rate. As the market interest rate is 10 percent and inflationary expectations are 4 percent, the real interest rate is approximately 6 percent. More precisely, using equation (6.13), the real interest rate is 5.77 percent.²⁹ Applying this real discount rate to the real annual costs and benefits yields an *NPV* equal to \$8,155. Thus, as long as no alternative equipment configuration offers a greater *NPV*, the city should purchase the larger trucks.

Using Nominal Dollars

If analysts take the nominal market interest rate facing the city as the appropriate discount rate, then they must predict future costs and benefits in nominal dollars. The right-hand column of Table 6.4 shows the anticipated benefits and costs of this project in nominal dollars, assuming a 4 percent annual inflation rate. As mentioned earlier, this example assumes that wage rates, gasoline prices, and other prices that figure into the benefit calculations increase at the same rate as the general price level. To convert amounts in real dollars to nominal dollars, simply inflate them by the expected rate of inflation, *m*. Notice that the city expects to receive \$233,972 when it sells the trucks at the end of the fourth year. This is called the *nominal liquidation value* of the trucks.

Discounting the benefits and costs measured in nominal dollars using a nominal (market) interest rate gives an *NPV* of the project equal to \$8,155, as shown in Table 6.4. Thus, the two methods give the same answer.

TABLE 6.4 The Net Present Value of Investment in New Garbage Trucks

<i>Year</i>	<i>Event</i>	<i>Annual Benefits and Costs (In real dollars)</i>	<i>Annual Benefits and Costs (In nominal dollars)</i>
0	Purchase	-500,000	-500,000
1	Annual savings	100,000	100,000
2	Annual savings	100,000	104,000
3	Annual savings	100,000	108,160
4	Annual savings	100,000	112,486
4	Liquidation	200,000	233,972
<i>NPV</i>		8,155 ^a	8,155 ^b

^aUsing a real discount rate of 5.769231 percent.

^bUsing a nominal discount rate of 10 percent.

RELATIVE PRICE CHANGES

The preceding section discusses how to handle expected price changes due to inflation (i.e., general price increases). It assumes relative prices do not change. This section discusses how to handle relative price changes.

The importance of relative price changes is well illustrated by a CBA of a coal development project in British Columbia to supply Japanese customers.³⁰ Consider Table 6.5. The second, third, and fourth columns contain the proposed project's benefits, costs, and net benefits, respectively, according to a CBA prepared by the provincial government of British Columbia (roughly equivalent to a state government in the United States).³¹ Overall, the net benefits were estimated to be \$330 million. The main beneficiaries of this project were expected to be the Canadian National Railway (CNR), which was the state-owned railway company at that time, and the Canadian federal

TABLE 6.5 CBA of North East Coal Development Project

	Benefits (\$ million)	Costs (\$ million)	Net Benefits		
			Base Case (\$ million)	90% Base Price (\$ million)	90% Base Price and Quantity (\$ million)
Mining Sector	3,316	3,260*	56 /	-146	-240
Transport Sector					
Trucking	33	33	0	0	0
Canadian National Railway	504	358	146	146	121
B.C. Railway	216	202	15	15	6
Port Terminal	135	150	-15	-15	-23
Analysis and Survey	11	11	0	0	0
British Columbia**					
Royalties	77	0	77	69	62
Corporate Taxes	154	0	154	125	107
Producer Surplus (Labor)	25	0	25	25	25
Environment	10	5	5	5	5
Highways***	0	88	-88	-88	-88
Tumbler Ridge Branchline	91	267	-176	-176	-185
Canada					
Corporate Taxes	132	0	132	107	92
Highways, Port Navigation	0	26	-26	-26	-26
Producer Surplus (Labor)	25	0	25	25	25
Totals	4,729	4,400	330	66	-119

* Includes taxes and royalties.

** Excluding impacts included elsewhere.

*** Highways, electric power, townsite.

Source: Based on W. G. Waters II, "A Reanalysis of the North East Coal Development," (undated), Tables 2 & 3. All figures in millions of 1980 dollars, assuming a 10 percent real discount rate with the discounting period ending in 2003 and no terminal value.

government, which would receive corporate taxes. Here corporate taxes and royalties are a benefit to government but a cost to the mining sector; thus, they are a transfer and they "net out." The British Columbian government was also expected to benefit from royalties and higher corporate taxes, but it would pay for the Tumbler Ridge Branchline, an extension of the provincially owned railway system to the North East Coal Development Project. The mining sector would benefit in terms of increased producer surplus. Also, this project was expected to create jobs for unemployed workers in British Columbia and the rest of Canada. The present value of the producer surplus to labor was estimated to be \$25 million. Notice that the analysis does not include any consumer surplus because all of the coal would be exported.

The fifth column contains the expected net benefits to each sector if the price of coal were to fall to 90 percent of the base price. Under this assumption, the aggregate net benefits would fall by \$264 million from \$330 million to \$66 million, a substantial change. The sixth column contains the expected net benefits to each sector if the price of coal were to fall to 90 percent of the base price and Japanese customers were to cut back their purchases of coal to 90 percent of their expected orders. Under this assumption, the overall net benefits would fall by \$449 million from \$330 million to -\$119 million.³² Thus, relatively small changes in relative prices and in quantities purchased have a huge impact on the *NPV* of this project.

In this example the benefits and costs are broken down by sector to illustrate distributional impacts. The main anticipated "winners" were the CNR and the federal government of Canada. If the price of coal fell by 10 percent, then the mining sector would lose money. Also, the residents of British Columbia would switch from being marginal "winners" to marginal "losers," largely because royalties and corporate taxes would decrease while the costs of highways and the Tumbler Ridge Branchline are fixed. If the price and quantity levels were to fall to 90 percent of the anticipated levels, then the mining sector would lose badly.

SENSITIVITY ANALYSIS IN DISCOUNTING

This chapter assumes that the rate that should be used to discount future benefits and costs is known. However, for reasons discussed in Chapter 10, there are significant differences of opinion about the correct social discount rate. Also, for reasons discussed earlier in this chapter, there is frequently considerable uncertainty about the magnitude of the terminal value. Because the discount rate and the terminal value often "drive" a CBA, analysts frequently conduct sensitivity analyses with respect to these two parameters.

Varying the Discount Rate and the Terminal Value

As discussed more fully in Chapter 7, the most straightforward way to perform sensitivity analysis is to systematically vary each parameter about which there is uncertainty and recalculate the *NPV*. This is very easy to do on a spreadsheet. If the policy recommendations are robust (i.e., the *NPV* remains either positive or negative) to plausible alternative values of the parameters, we can have greater confidence in them.

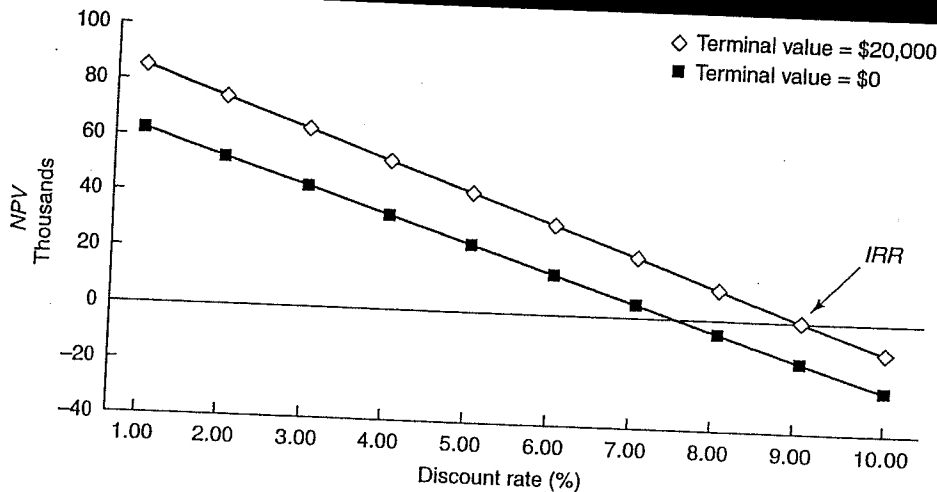
Figure 6.7 plots the *NPV* of the library project against the discount rate for two different terminal values, one with a terminal value of \$20,000 and the other with a terminal value of \$0. Clearly, the choice of the discount rate is important. As the discount rate increases, the *NPV* of the project decreases. This common pattern arises for investment projects whose costs occur early and whose benefits occur late. Using a higher discount rate results in a lower *NPV* because the future benefits are discounted more than are the more immediate costs.

The top curve corresponds to a terminal value of \$20,000. The *NPV* equals \$17,276 if the discount rate equals 7 percent. The *break-even* discount rate is 8.9 percent, which can be read off the graph where the *NPV* = 0 or found exactly by trial and error.³³ As long as the appropriate discount rate is less than 8.9 percent, the project offers a positive *NPV* and should be adopted. If the appropriate discount rate is more than 8.9 percent, then the project has a negative *NPV* and it should not be approved. Most analysts agree that the appropriate real social discount rate is less than 8.9 percent—see Chapter 10. Consequently, we can be reasonably confident that if the project goes ahead it will have a positive *NPV*.

If the terminal value is zero, the curve shifts down by the discounted value of \$20,000. As this value decreases as the discount rate increases, the curve shifts down less for high interest rates than for low interest rates. Thus, it is flatter than the curve with a terminal value of \$20,000. Although the *NPV* is smaller at every interest rate, it is still positive as long as the social discount rate is less than 7.35 percent. As most analysts believe that the appropriate real social discount rate is less than 7.35 percent, we would still recommend the project should proceed.

Of course, we can also compute the *break-even terminal value*—the terminal value at which the *NPV* equals zero. Assuming the appropriate discount rate is 7 percent, the *break-even* terminal value is -\$4,230, which implies the city would just break even on the project if it cost \$4,230 to dismantle the project at the end.³⁴

FIGURE 6.7 Sensitivity Analysis: *NPV* of the Library Information System as a Function of the Discount Rate



CC

AP
SI
P

The Internal Rate of Return

The discount rate at which the *NPV* is zero is also called the *internal rate of return (IRR)*. The *IRR* of the library information system is 8.9 percent, which implies that it is equivalent to a project of similar size that provides annual benefits equal to 8.9 percent of the original amount for five years (the length of the project) and returns all of the initial invested capital at the end of the 5th year.

The *IRR* can be used to express the decision rule for selecting projects *when there is only one alternative to the status quo*. If the *IRR* of a project is greater than the appropriate discount rate, then one should proceed with the project; if the *IRR* is less than the appropriate discount rate, one should not proceed with it. In this example, the library should proceed with the project because the *IRR* of 8.9 percent is greater than the appropriate discount rate of 7 percent. The basic idea, which we discuss in depth in Chapter 10, is that society should only invest in projects that earn a higher return than could be earned by investing the resources elsewhere. In other words, the appropriate discount rate should reflect the opportunity cost of the funds.

There are, however, a number of potential problems with using the *IRR* for decision making. First, it may not be unique; that is, there may be more than one discount rate at which the *NPV* is zero. This problem only arises when annual net benefits change more than once from positive to negative (or vice versa) during the discount period. Second, *IRRs* are percentages (i.e., ratios), not dollar values. Therefore, they should not be used to select one project from a group of mutually exclusive projects that differ in size. This scale problem always arises with the use of ratios, including *IRRs* and benefit-cost ratios. Nonetheless, if it is unique, the *IRR* conveys useful information to decision makers or other analysts who want to know how sensitive the results are to the discount rate.³⁵

CONCLUSION

This chapter presents the main issues concerning the mechanics of discounting in CBA. It assumes that the appropriate discount rate is known. In fact, determination of the appropriate discount rate to use in CBA is a contentious issue, which we discuss in Chapter 10.

Appendix 6A

SHORTCUT METHODS FOR CALCULATING THE PRESENT VALUE OF ANNUITIES AND PERPETUITIES

In many practical situations the benefits or costs of a project can be treated as annuities or perpetuities. The present value of an annuity or perpetuity is relatively easy to compute.

An *annuity* is an equal, fixed amount received (or paid) each year for a number of years. A *perpetuity* is an annuity that continues indefinitely. Suppose, for example, that in order to finance a new highway, a state government issues \$100 million worth of 30-year

bonds with an interest rate of 7 percent paid annually. The annual interest payments of \$70,000 are an annuity. If, at the end of *each* 30-year period, the state government refinances the debt by issuing another 30-year bond that also has an interest rate of 7 percent, then the annual interest payments of \$70,000 would continue indefinitely, which is a perpetuity. Sometimes an annuity or perpetuity grows or declines at a constant rate. In any of these situations, using equation (6.6) or (6.7), the method described in the main body of this chapter to compute the present value, can be extremely time consuming. Fortunately, some simple formulas enable analysts to compute present values easily.

PRESENT VALUE OF AN ANNUITY

The library information system problem contains two annuities: the annual benefits of \$100,000 per year for five years, which we refer to as annuity A1, and the annual costs of \$20,000 per year for five years, which we refer to as annuity A2. From Figure 6.4 we see that the present value of A1 is \$410,020 and the present value of A2 is \$82,004. But there is an easier way to obtain the present values.

Using equation (6.6), the present value of an annuity of A per annum (with payments received at the end of each year) for n years with interest at i percent is given by:

$$PV = \sum_{t=1}^n \frac{A}{(1+i)^t}$$

This is the sum of n terms of a geometric series with the common ratio equal to $1/(1+i)$. Consequently:

$$PV = A \times a_i^n \tag{6A.1}$$

where:

$$a_i^n = \frac{1 - (1+i)^{-n}}{i} \tag{6A.2}$$

The term, a_i^n , which equals the present value of an annuity of \$1 per year for n years when the interest rate is i percent, is called an *annuity factor*. Tables of annuity factors are contained in most finance textbooks and are also built into many handheld calculators and computer spreadsheets.

Returning to our library example, the present value of annuity A1 computed using equations (6A.1) and (6A.2) is:

$$PV(A1) = \$100,000 \times \frac{1 - (1 + 0.07)^{-5}}{0.07}$$

$$PV(A1) = \$100,000 \times 4.1002$$

$$PV(A1) = \$410,020$$

Similarly:

$$PV(A2) = \$20,000 \times 4.1002 = \$82,004$$

Although this example deals with only a five-year annuity, it is easy to compute the present value of annuities that extend over much longer periods.

When working with annuities it is important to get the timing of the cash flows exactly right. Equation (6A.1) assumes that the benefits or costs occur at the end of each year, with the first payment occurring one year from now. This type of annuity is called an *ordinary annuity*. Care is required when the annuity payments start now or when they begin more than one year from now.

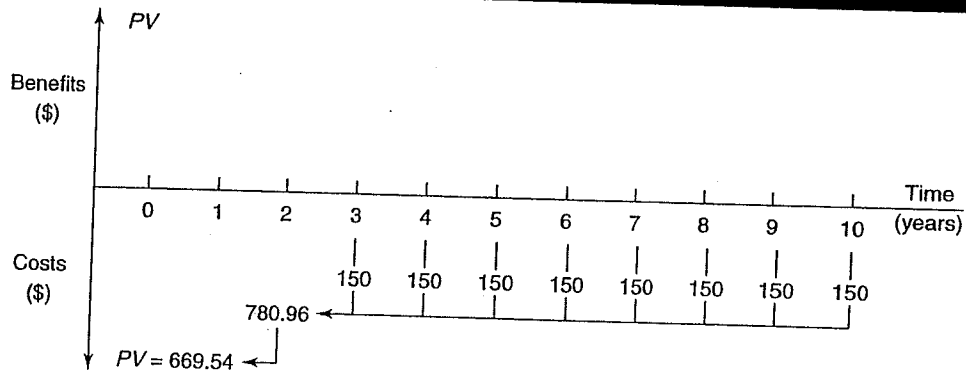
An *annuity due* is an annuity with payments that occur at the beginning of each year. Many spreadsheets allow one to compute the present values of ordinary annuities and of annuities due. If the spreadsheet (or calculator) computes only ordinary annuities, then one can calculate the *PV* of an annuity due for n years by computing the *PV* of an ordinary annuity for $n - 1$ years and adding the value of one initial payment or receipt made today.

A *deferred annuity* is an annuity whose first payment is deferred until after the first year. Suppose, for example, a government agency is considering refinancing some of its debt. Currently, it is scheduled to make debt payments of \$150,000 per year for seven years with the first payment in three years. Assuming interest rates are 8 percent, what is the present value of this obligation? A time line for this problem is shown in Figure 6A.1. The first step is to treat the seven annual payments as an ordinary annuity and compute its present value using equations (6A.1) and (6A.2) or a calculator. The *PV* of the annuity is \$780,956. The second and most important step is to recognize that this amount is the value of an ordinary annuity *in two years' time*—one year before payments are scheduled to begin. The third and final step is to discount the \$780,956 back two years to obtain the present value of \$669,543.

It is informative to note how the present value of an annuity varies with time and the interest rate. *The present value of an annuity decreases as interest rates increase, and vice versa.* This is a partial explanation for why bond prices rise as interest rates fall, as they did in the United States between 1988 and 1993.

Another important observation is that when there is a relatively constant annuity stream, *annuity payments received after about the twentieth year add little to the present value when interest rates are 10 percent or higher.* Thus, private companies are often reluctant to make very long-term investments such as reforestation.

FIGURE 6A.1 Present Value of \$150 per Year (in Thousands) for Seven Years, Starting in Three Years



PRESENT VALUE OF A PERPETUITY

A *perpetuity* is an annuity that continues indefinitely. Taking the limit of equation (6A.2) as n goes to infinity, the annuity factor reduces to $1/i$, if $i > 0$. Consequently, the present value of an amount, denoted by A , received (at the end of) each year in perpetuity is given by:

$$PV = \frac{A}{i} \quad \text{if } i > 0 \quad (6A.3)$$

To provide some intuition for this formula, suppose that a municipality has an endowment of \$10 million. If interest rates are 6 percent, then this endowment will provide annual interest payments of \$600,000 indefinitely. More generally, if the municipality has an endowment of X and if the interest rate is i , then the perpetual annual income from the endowment, denoted by A , is given by $A = iX$. Rearranging this equation, the present value of the perpetual annuity is given by $X = A/i$, which is equation (6A.3).

Equation (6A.3) is easy to apply. For example, the present value of a perpetuity of \$150,000 per year when interest rates are 8 percent is:

$$PV = \frac{\$150,000}{0.08} = \$1,875,000$$

When interest rates are 10 percent, the present value of a perpetuity is especially easy to calculate: It equals the perpetuity multiplied by 10. For example, the present value of a perpetuity of \$150,000 per year is \$1,500,000 when interest rates are 10 percent.

THE PRESENT VALUE OF AN ANNUITY THAT GROWS OR DECLINES AT A CONSTANT RATE

Sometimes a project's benefits (or costs) grow at a constant rate. Let B_t denote the benefits in year t . If the annual benefits grow at a constant rate, g , then the benefits in year t will be:

$$B_t = B_{t-1}(1 + g) = B_1(1 + g)^{t-1} \quad t = 2, \dots, n \quad (6A.4)$$

Under these circumstances, and if $i > g$, then the present value of the total benefits—the stream over n years—can be shown to be:¹

$$PV(B) = \frac{B_1}{(1 + g)} \times a_{i_0}^n, \quad (6A.5)$$

where, $a_{i_0}^n$ is defined by equation (6A.2) and:

$$i_0 = \frac{i - g}{1 + g} \quad (6A.6)$$

Comparing equation (6A.1) with (6A.5) shows that the PV of a benefit stream that starts at B_1 in year 1 and grows at a constant rate g for $n - 1$ additional years, when the interest rate is i , equals the PV of an annuity of $B_1/(1 + g)$ for n years when the interest rate is i_0 , where i_0 is given by equation (6A.6).

To illustrate how to calculate the present value of a stream of benefits that grows at a constant rate, return again to the library example. Suppose we now assume that, due to increased use, annual benefits grow at 2 percent per annum (and, as before, $i = 0.07$). From equation (6.A.5), the present value of the stream of benefits equals the present value of an annuity of $\$100,000/1.02 = \$98,039$ per annum for five years, discounted at the following rate:

$$i_0 = \frac{i - g}{1 + g} = \frac{0.07 - 0.02}{1.02} = 0.049$$

which amounts to $\$425,636$.² In contrast, when annual benefits are constant ($\$100,000$), the present value equals $\$410,020$, a difference of $\$15,616$. This increase would be carried straight to the bottom line as the *NPV* would also increase by $\$15,616$; in fact, it almost doubles from $\$17,276$ to $\$32,892$. This example illustrates that even quite small growth rates can have large impacts on *NPVs*.

If the growth rate is small, then $B_1/(1 + g) \approx B_1$ and $i_0 \approx i - g$. Therefore, from equation (6A.5), the present value of a benefits stream that starts at B_1 and grows at rate g for $n - 1$ additional years approximately equals the present value of an annuity of B_1 for n years discounted at rate $i - g$. This approximation makes it clear that when benefits grow at a positive rate, the annuity is discounted at a lower rate, which will yield a higher *PV*. On the other hand, if the benefits are declining at a constant rate, then the annuity is discounted at a higher rate, which will yield a lower *PV*.

To illustrate this approximation, note that the present value of a stream of benefits that starts at $\$100,000$ per year and grows at 2 percent per annum for four additional years with $i = 0.07$ is approximately equal to the present value of an annuity of $\$100,000$ per year for five years discounted at $i - g = 5$ percent, which equals $\$432,949$.³ This value is slightly higher than the correct amount of $\$425,634$, but it is within 2 percent of the right answer and it is easier to calculate.

Equation (6A.5) only holds if the interest rate exceeds the growth rate: $i > g$. If $i \leq g$, then it should not be used. Importantly, though, it can be used if g is negative, that is, if benefits decline at a constant rate. Of course, equation (6A.5) pertains to costs that change at a constant rate as well as to benefits that change at a constant rate.

PRESENT VALUE OF BENEFITS (OR COSTS) THAT GROW OR DECLINE AT A CONSTANT RATE IN PERPETUITY

If initial benefits, B_1 , grow indefinitely at a constant rate g and if the interest rate equals i , then the *PV* is found by taking the limit of equation (6A.5) as n goes to infinity, which gives:

$$PV(B) = \frac{B_1}{i - g}, \quad \text{if } i > g \quad (6A.7)$$

Some finance students will recognize this model as the Gordon growth model, which is also called the dividend growth model. This model can be used to value a stock that yields a constant flow of dividends that grow at a constant rate.⁴ As before, this formula holds only if $i > g$.

Appendix 6B

DISCOUNTING WITH MULTIPLE COMPOUNDING
IN EACH PERIOD**FUTURE VALUE WITH MULTIPLE COMPOUNDING IN A YEAR**

Thus far we have assumed that interest is calculated only once each period, with a period being a year. In practice, mortgages, savings accounts, and other investments compound interest more frequently than once a year.

Suppose we can invest \$1,000, the annual interest rate is 8 percent, but interest is compounded semiannually, that is, every six months. How much will we have at the end of the year? With an annual interest rate of 8 percent but with semiannual compounding, we earn 4 percent interest every six months. Therefore, at the end of the first six-month period, we will have:

$$\$1,000(1 + 0.04) = \$1,000(1.04) = \$1,040$$

If we leave the money in the bank for another six months, at the end of the year we will have:

$$\$1,040(1.04) = \$1,081.60$$

Clearly, investing for one year at 8 percent annual interest with semiannual compounding is equivalent to investing for two years at 4 percent per annum with annual compounding:

$$\$1000 \left(1 + \frac{0.08}{2}\right)^2 = \$1,000(1.04)^2 = \$1,081.60$$

When interest is compounded semiannually, there are twice as many periods as before. However, *during each period* the interest rate is only half the annual rate.

A comparison of the amount realized with semiannual compounding to the amount obtained with annual compounding shows that as interest is compounded more frequently, the future value increases more quickly. This occurs because as interest is compounded more frequently, interest on the interest is earned sooner. Thus, the effective annual interest rate with multiple compounding is higher than the effective annual rate with single compounding. In this example, the *effective annual interest rate*, which is the interest rate that one would have to obtain if interest were compounded annually in order to yield the same amount as multiple compounding, is 8.16 percent.

In general, the future value of X in one year at interest rate i compounded k times a year is:

$$FV = X \left(1 + \frac{i}{k}\right)^k$$

Exercises for Chapter 6

1. A highway department is considering building a temporary bridge to cut travel time during the three years it will take to build a permanent bridge. The temporary bridge can be put up in a few weeks at a cost of \$740,000. At the end of three years, it would be removed and sold for scrap at a net cost of \$81,000, estimated as the net cost of a similar operation done today. Based on estimated time savings and wage rates, fuel savings, and reductions in risks of accidents, department analysts predict that the benefits in real dollars would be \$275,000 during the first year, \$295,000 during the second year, and \$315,000 during the third year. Departmental regulations require use of a real discount rate of 6 percent.
 - a. Calculate the present value of net benefits assuming that the benefits are realized at the end of each of the three years.
 - b. Calculate the present value of net benefits assuming that the benefits are realized at the beginning of each of the three years.
 - c. Calculate the present value of net benefits assuming that the benefits are realized in the middle of each of the three years.
 - d. Calculate the present value of net benefits assuming that half of each year's benefits are realized at the beginning of the year and the other half at the end of the year.
 - e. Does the temporary bridge pass the net benefits test?
2. A government data processing center has been plagued in recent years by complaints from employees of back pain. Consultants have estimated that upgrading office furniture at a net cost of \$425,000 would reduce the incidence and severity of back injuries, allowing the center to avoid medical care that currently costs \$68,000 each year. They estimate that the new furniture would also provide yearly benefits of avoided losses in work time and employee comfort worth \$18,000. The furniture would have a useful life of five years, after which it would have a positive salvage value equal to 10 percent of its initial net cost. The consultants made their estimates of avoided costs assuming that they would be treated as occurring at the beginning of each year. In its investment decisions, the center uses a nominal discount rate of 9.5 percent and an assumed general inflation rate of 4 percent. It expects the inflation rate for medical care to run between 4 percent and 6 percent but is uncertain as to the exact rate. In other words, it is uncertain as to whether the cost of medical care will inflate at the same rate as other prices or rise 2 percent faster. Should the center purchase the new furniture?
3. A town's recreation department is trying to decide how to use a piece of land. One option is to put up basketball courts with an expected life of eight years. Another is to install a swimming pool with an expected life of 24 years. The basketball courts would cost \$180,000 to construct and yield net benefits of \$40,000 at the end of each of the eight years. The swimming pool would cost \$2.25 million to construct and yield net benefits of \$170,000 at the end of each of the 24 years. Each project is assumed to have zero salvage value at the end of its life. Using a real discount rate of 5 percent, which project offers larger net benefits?
4. The environmental protection agency of a county would like to preserve a piece of land as a wilderness area. The current owner has offered to lease the land to the county for 20 years in return for a lump-sum payment of \$1.1 million, which would be paid at the beginning of the 20-year period. The agency has estimated that the land would generate \$110,000 per year in benefits to hunters, bird watchers, and hikers. Assume that the lease price represents the social opportunity cost of the land and that the appropriate real discount rate is 6 percent.

- a. Assuming that the yearly benefits, which are measured in real dollars, accrue at the end of each of the 20 years, calculate the net benefits of leasing the land.
 - b. Some analysts in the agency argue that the annual real benefits are likely to grow at a rate of 2 percent per year due to increasing population and county income. Recalculate the net benefits assuming that they are correct.
5. Imagine that the current owner of the land in the previous exercise was willing to sell the land for \$2 million. Assuming this amount equaled the social opportunity cost of the land, calculate the net benefits if the county were to purchase the land as a permanent wildlife refuge. In making these calculations, first assume a zero annual growth rate in the \$110,000 of annual real benefits; then assume that these benefits grow at a rate of 2 percent per year.

Notes

1. Parts of this chapter draw on Anthony E. Boardman and David H. Greenberg, "Discounting and the Social Discount Rate," in *Handbook of Public Finance*, Fred Thompson and Mark T. Green, eds. (New York: Marcel Dekker, 1998), pp. 269–318.
2. A T-bill is a short-term bond issued by the Treasury Department of the U.S. government. Usually, the minimum size is \$100,000. It yields an interest rate that is slightly higher than the rate of interest offered by banks on personal savings accounts. Some banks offer a T-bill account, which enables customers with less than \$100,000 (but more than some specified minimum) to earn almost the same rate as that given on T-bills.
3. $PV = \frac{\$10,280,374}{1.07} = \$10,000,000$
4. The following equation summarizes the relationship between discounting and compounding for projects of one year's duration:

$$PV = \frac{FV}{1+i}$$
5. At the end of the first year one would have $FV = X(1+i)$. At the end of the second year, one would have $FV = [X(1+i)](1+i) = X(1+i)^2$, and so on.
6. One should be aware, however, that different sources may give slightly different answers due to rounding error. For example, the more accurate answer to the foregoing question, without rounding, is \$13,107,960. In practice, rounding errors are not material.
7. The \$20,000 figure includes the dismantling costs.
8.
$$NPV = \sum_{t=0}^n \frac{B_t}{(1+i)^t} - \sum_{t=0}^n \frac{C_t}{(1+i)^t}$$

$$= \sum_{t=0}^n \frac{B_t - C_t}{(1+i)^t} = \sum_{t=0}^n \frac{NB_t}{(1+i)^t}$$
9. Comparing equation (6.11) with equation (6.9) implies:

$$T(k) = NPV - \sum_{t=0}^k \frac{NB_t}{(1+i)^t} = \sum_{t=k+1}^n \frac{NB_t}{(1+i)^t}$$
10. Terminal values are well understood in the context of the private-sector capital budgeting literature. See, for example, Peter Lusztag and Bernhard Schwab, *Managerial Finance in a Canadian Setting* (Toronto, Ontario: Butterworths, 1988), Chapter 8: Capital Budgeting, especially pp. 310–311. See also pp. 976–979.
11. For example,

$$NPV = \sum_{t=0}^{20} \frac{NB_t}{(1+i)^t} + T(20) = \sum_{t=0}^{30} \frac{NB_t}{(1+i)^t} + T(30)$$
12. For a discussion about how to measure depreciated capital, see Charles R. Hulten, "The Measurement of Capital," Chapter 4 in *Fifty Years of Economic Measurement: The Jubilee of the Conference on Research in Income and Wealth*, Ernst R. Berndt and Jack E. Triplett, eds. (Chicago: The University of Chicago Press, 1990), pp. 119–152.
13. If project A were two-thirds the length of project B, then the analyst should compare three project As back-to-back with two project Bs back-to-back.
14. The difference between the NPV of five back-to-back cogeneration plants and the NPV of only one cogeneration plant is \$10.94 million. We would have arrived at the same figure if we had evaluated building only one cogeneration plant but assigned a terminal value of \$10.94 million to this project.
15. The annuity factor for the hydroelectric project is the present value of an annuity of \$1 per year for 75 years using an interest rate of 8 percent, which, using equation (6A.2) or a calculator, equals

- 12.4611. Similarly, the annuity factor for the co-generation project equals the present value of \$1 per annum for 15 years at an interest rate of 10 percent.
16. A broader measure is the implicit deflator for gross national product (GNP), which is the ratio of GNP measured at current prices to GNP measured at prices in some base year. Whereas CPI is based on a standard market basket of consumer goods, the implicit deflator for GNP is a comprehensive price index.
17. The most recent CPI figures for the United States as well as CPI figures back to 1913 are available from the Bureau of Labor Statistics at its Web site: www.bls.gov. The most recent CPI figures for Canada are available from Statistics Canada at its Web site: www.statcan.ca. For long-term historical Canadian data, see F. H. Leacy, ed., *Historical Statistics of Canada* (Ottawa, Ontario: Statistics Canada, 1983).
18. See Michael J. Boskin, Ellen R. Dulberger, Robert J. Gordon, Zvi Griliches, and Dale W. Jorgenson, *Toward a More Accurate Measure of the Cost of Living*, Final Report to the Senate Finance Committee from the Advisory Committee to Study the Consumer Price Index (Washington, DC: Senate Finance Committee, 1996). In Canada, the CPI was overestimated by about 0.7 percentage points per annum.
19. For a recent summary, see Brent R. Moulton, "Bias in the Consumer Price Index: What Is the Evidence?" *Journal of Economic Perspectives*, 10, no. 4 (1996), 159–177.
20. See Zvi Griliches and Iain Cockburn, "Generics and New Goods in Pharmaceutical Price Indexes," *American Economic Review*, 84, no. 5 (1994), 1213–1232.
21. For more information about the CPI revisions see <http://stats.bls.gov/cpihome.htm>.
22. To convert the \$100,000 in 1994 dollars into an amount in 1999 dollars, we divide by the CPI for 1994 and multiply by the CPI for 1999: $(\$100,000/148.2) \times 166.6 = \$112,416$. Between 1994 and 1999, the CPI increased by 2.34 percentage points per annum, on average. If the actual CPI increased by one percentage point per annum less than the reported CPI, it would have increased by 1.34 percentage points per annum, from 148.2 to 158.62. Consequently, the \$100,000 in 1994 would equal $(\$100,000/148.2) \times 158.6 = \$107,018$ in 1999 dollars.
23. Recall that the NPV for a project is given by equation (6.8), and suppose that these benefits, B_t , and costs, C_t , are in nominal dollars and i is the nominal interest rate. Let b_t denote real benefits, c_t denote real costs, and suppose the rate of inflation is m , then, using equation (6A.4), $B_t = b_t(1+m)^t$ and $C_t = c_t(1+m)^t$.
- Consequently:
- $$NPV = \sum_{t=0}^n \frac{B_t - C_t}{(1+i)^t} = \sum_{t=0}^n \frac{(b_t - c_t)(1+m)^t}{(1+i)^t}$$
- Now, setting $1/(1+r) = (1+m)/(1+i)$, where r is the real interest rate, gives:
- $$NPV = \sum_{t=0}^n \frac{b_t - c_t}{(1+r)^t}$$
24. The expected CPI in t periods in the future equals $(1+m)^t$ times the current CPI. Therefore, dividing the future amounts measured in nominal dollars by $(1+m)^t$, in accordance with equation (6.5), is exactly the same as the method implied at the end of the previous subsection for converting amounts expressed in year b dollars into amounts expressed in year a dollars, namely, dividing by the expected CPI in t periods in the future and multiplying by the CPI for the current year.
25. This relationship is known as the Fisher effect. For derivation of this expression, rewrite the equation that introduces r in note 23. Alternatively, note that \$1 invested today at a nominal interest rate of i yields $\$(1+i)$ one year later. However, with an inflation rate of m during the year, $\$(1+i)$ one year from now buys only as much as $\$(1+i)/(1+m)$ does today. The real interest rate, r , is, therefore, defined by $\$(1+r) = \$(1+i)/(1+m)$. Rearranging this expression gives equation (6.13).
26. An alternative procedure draws on the Fisher model and assumes the real yield on bonds of a particular duration is constant over time. Specifically, this estimate of expected inflation during the life of the project equals the current yield on bonds that have the same term as the project minus the historical real yield on bonds of this term. For example, if the life of the project is 20 years, if the yield on bonds that mature in 20 years is currently 10 percent, and if 20-year bonds typically yield a real return of 2 percent, then the expected annual rate of inflation over this period is about 8 percent. If the life of the project is only

- three years, if the yield on bonds that mature in three years is 9 percent, and if the historical real yield on bonds that mature in three years is 1.5 percent, then the expected annual rate of inflation over the next three years is about 7.5 percent. Notice that in this example, the market expects inflation will be higher in years 4 to 20 than in the first three years. For historical nominal and real returns on stocks, corporate bonds, government bonds, and treasury bills, see Ibbotson Associates, *Stocks, Bonds and Inflation: 1989 Yearbook* (Chicago: Ibbotson Associates, 1989), and their more recent publications.
27. For more information about the Livingston survey and the SPF, see the Web site of the Federal Reserve Bank of Philadelphia at <http://phil.frb.org>. For more information about the Michigan survey of households, see the Institute for Social Research's Survey of Consumer Attitudes Web site at <http://athena.sca.isr.umich.edu>.
 28. Lloyd B. Thomas Jr., "Survey Measures of Expected U.S. Inflation," *Journal of Economic Perspectives*, 13, no. 4 (1999), 125-144.
 29. From equation (6.13), $r = (0.10 - 0.04)/(1 + 0.04) = 5.77$ percent. More accurately, the real discount rate is 5.769231 percent, which is the rate used in the *NPV* calculations in this example in order to ensure that the results for the real and nominal methods are identical.
 30. An initial study was prepared by the government of British Columbia, "A Benefit-Cost Analysis of the North East Coal Development" (Victoria, BC: Ministry of Industry and Small Business, 1982). The sensitivity analyses were prepared by W. W. Waters II, "A Reanalysis of the North East Coal Development" (Working paper, University of British Columbia, undated).
 31. The initial study made shadow price adjustments for foreign exchange earnings. These are excluded from Table 6.5 because, according to Bill Waters, the author of the reanalysis, they are probably inappropriate.
 32. This scenario of declining prices and declining quantities sold is quite plausible for this project. The major Japanese customers were simultaneously encouraging development of Australian and other sources. If these sources came on line at the same time, there would be a worldwide excess supply, internationally determined coal prices would fall, and demand for BC coal would fall.
 33. By definition, the break-even discount rate can be found by setting the left-hand side of equations (6.8) or (6.9) equal to zero and solving for i . In practice, this can be quite difficult. Usually, trial and error is much easier. Some spreadsheet programs allow this to be done quite readily.
 34. The *NPV* is \$17,276, which has a future value in year 5 of \$24,230. Thus, one would obtain an *NPV* equal to zero if the terminal value were reduced by \$24,230 from \$20,000 to -\$4,230.
 35. If expenditures equal social costs and if the total amount of expenditures is constrained, then ranking projects on the basis of the *IRR* criterion may maximize the total *NPVs*. For example, if three projects cost a total of \$1 billion and, in addition, each has a higher *IRR* than a fourth project that costs \$1 billion by itself, then the first three projects will also have a larger *combined NPV* than the *NPV* of the fourth project. In effect, the *IRR* provides an estimate of the average annual net benefit per unit of (constrained) expenditure. Problems with ranking projects in terms of their *IRRs* only arise when the total expenditure on the projects with the highest *IRRs* does not exactly equal the total amount available. Even if the first, second, and third projects have higher *IRRs* than the fourth project, the fourth project *may* still have a higher *NPV* than the three smaller projects combined if its total cost is larger (but still no more than \$1 billion). In any constrained project choice setting, the optimal set of projects can always be found readily by using linear programming.

Appendix Notes

$$1. PV(B) = \sum_{t=1}^n \frac{B_1(1+g)^{t-1}}{(1+i)^t}$$

$$= \sum_{t=1}^n \frac{B_1}{(1+g)} \times \left(\frac{1+g}{1+i}\right)^t$$

Setting $i_0 = (i - g)/(1 + g)$ implies $1/(1 + i_0) = (1 + g)/(1 + i)$. Consequently:

$$PV(B) = \sum_{t=1}^n \frac{B_1}{(1+g)} \times \frac{1}{(1+i_0)^t} = \frac{B_1}{(1+g)} \times a_{i_0}^n$$

2. Alternatively, substituting equation (6A.2) into equation (6A.5) gives the following formula for computing the *PV* of the benefits:

$$PV(B) = \frac{B_1}{1+g} \times \frac{1 - (1+i_0)^{-n}}{i_0}$$

Therefore, for this example:

$$\begin{aligned} PV(B) &= \frac{100,000}{(1+0.02)} \times \frac{1 - (1+0.049)^{-5}}{0.049} \\ &= \$98,039 \times 4.3415 = \$425,636 \end{aligned}$$

3. That is:

$$\begin{aligned} PV(B) &= \sum_{t=1}^n \frac{B_1}{(1+i-g)^t} \\ &= \sum_{t=1}^5 \frac{100,000}{(1+0.07-0.05)^t} = \$432,948 \end{aligned}$$

4. See M. Gordon, *The Investment, Financing and Valuation of the Corporation* (Homewood, IL: Richard D. Irwin, 1962), pp. 46-47.
5. Formally, $e = \lim(1 + 1/n)$ as $n \rightarrow \infty$.