Renegotiation-Proof Contracts in Repeated Agency

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Abstract

Renegotiation-proof contracts are studied in infinitely repeated principal-agent contracting. Contracts satisfying a weaker notion of renegotiation-proofness always exist. With risk neutrality, efficient full-commitment contracts are renegotiation-proof if the agent's expected lifetime utility is above a critical level; otherwise or if the agent is risk averse then efficient full-commitment contracts may not be renegotiation-proof. The renegotiation-proof value function has a simple characterization: it is the optimal value function with an appropriate lower bound placed on the agent's expected lifetime utility. Sufficient conditions are provided for renegotiation-proof value functions in finite horizon setting to converge to a renegotiation-proof value function in infinite horizon setting, as time goes to infinity.

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1 Introduction

Consider a principal who hires an agent to work on a project for a long period of time. The principal may offer the agent wage contracts that span part or the whole life of the employment relationship. The main issue that arises in this long-term principal-agent contracting problem concerns commitment. With full commitment, the principal can offer long-term contract to the agent at the beginning of the employment and sticks to the contract throughout the life of the relationship. With limited commitment, the principal and the agent may agree on and bind themselves to short-term contracts (for example, one-period contracts), but can not commit themselves to any future contracts. The third form is long-term contracts with renegotiation: the parties can write long-term contract but may renegotiate and change the contract in the future as long as there is mutual consent. Although the commitment issue has been extensively analyzed within *finite-horizon* framework,[1] it has not been the case for infinite-horizon applications where the durations of relationships are not known beforehand. The dynamic contracting literature focuses on full-commitment long-term contracting and on limited commitment contracting,[2] but offers only limited treatment of long-term contracting with renegotiation, which is the subject matter of this paper. Specifically, this paper investigates the properties of long-term renegotiation-proof contracts and their connections with limited and full commitment contracts in the infinite-horizon principal-agent setting.

Infinite-horizon applications present new challenges that are absent in finite-horizon settings. At a conceptual level, renegotiation-proofness is well understood in finite horizon (see Wang (2000)). Consider finitely repeated principal-agent contracting where the principal and the agent can renegotiate the remaining contract at the beginning of each period. If there is only one period, renegotiation-proofness amounts to ex ante Pareto optimality. If there are more than one but finitely many periods, renegotiation-proofness can be formulated as follows: Start-

[1] Fudenberg, Holmstrom, and Milgrom [7] and Rey and Salanie [18] examine connections between short-term and long-term contracts. Wang [25] investigates long-term renegotiation-proof contracts.

[2] The literature starts with the work of Spear and Srivastava [22] and that of Green [9]. See Phelan [15], Thomas and Worrall [23] for contracting with limited commitment. Ljungqvist and Sargent [13] offer a comprehensive treatment of many applications of dynamic contracting.

ing from the final period, one-period renegotiation-proof contracts are just Pareto optimal contracts. Using backward induction, T-period renegotiation proof contracts are Pareto optimal contracts subject to the constraint that their T-1-period continuation contracts are renegotiation proof. This backward induction procedure, however, will not work for infinite-horizon applications, which is the main reason why so many competing concepts of renegotiation-proofness have been proposed for general infinitely repeated games.[3]

Here I posit two intuitive axioms as basic requirements for renegotiation-proof contracts, which are natural extensions from finite-horizon setting. The first axiom (called *recursion*) requires that a contract be renegotiation proof if and only if every continuation contract is renegotiation proof (henceforth RP). The second axiom (called *pareto optimality*) requires that a RP contract not be Pareto dominated by any other RP contract. The set of contracts that meet the two conditions satisfy *Self-Pareto-Generating*: the Pareto frontier of the principal's optimal value function is identical to the continuation value function (see Section 4). This notion of renegotiation-proofness coincides with the concept of internal renegotiation-proofness put forward by Ray [17]. In the analysis of this paper, the principal is permitted to publicly randomize over menu of contracts, which guarantees existence of contracts that satisfy a weaker notion of renegotiation-proofness and facilitates the derivation of characterization results.

The key condition that determines the impact of renegotiation is the extent of punishment the principal can inflict on the agent in a single period. If the principal can arbitrarily lower the agent's lifetime utility by simply reducing the agent's income in a single period (which is possible if the agent's utility from income is unbounded from below), then efficient full-commitment contracts are renegotiation-proof: every continuation contract of every ex ante Pareto efficient contract is itself efficient. This result is related to earlier studies of finite-horizon applications. Rey and Salanie [18] show that short-term (two-period) contracts with renegotiation can achieve full-commitment long-term efficiency if inter-period transfers are unlimited (surjectivity) and agents' objectives are conflicting. Fudenberg, Holmstrom and Milgrom [7] identify conditions that guarantee one-period

[3]See among others, Bernheim and Ray [4], Farrell and Maskin [6], van Damme [24], Abreu, Pearce, and Stacchetti [1], Bergin and MacLeod [3], Ray [17], and Kocherlakota [11]. Bergin and MacLeod [3] also discuss the relationships among various concepts.

contracts are sufficient for achieving long-term efficiency. Their conditions include: there is common knowledge about preferences and technology at all renegotiation stages; both agents have equal access to credit market (so the agents are effectively risk neutral toward income streams); the utility frontier at every history generated by the set of incentive compatible continuation contracts is downward sloping. In comparison, in the current model the agent may not have access to credit market and may be risk averse. Moreover, unlimited punishment falls short of the surjectivity condition (which also requires unlimited reward so in effect requires agent's utility function to be full range) of Rey and Salanie. As a result, renegotiation-proofness of the full-commitment contracts does not imply implementation by short-term contracts.

The more interesting situation occurs when the principal can not inflict unlimited punishment on the agent in a single period. This can happen if the agent's utility from income is bounded from below or if there is limited liability on the part of the agent which requires a minimum wage. An efficiency-wage type of argument applies (See Shapiro and Stiglitz [21]). Renegotiation will have a bite. The reason is as follows. Because the principal has to offer the agent a minimum level of instantaneous utility at any point in time, the agent will normally receive positive rent if he is expected to exert nontrivial effort, namely the agent must be paid an expected utility over and above his reservation utility. In dynamic context, the principal can structure the agent's intertemporal rent payments to maximize her expected profits. I identify two properties of the efficient full-commitment rent structure. The first is the use of ex post Pareto inefficient continuation contracts as punishment device. The second is the use of deferred payment as reward device. These properties of the full-commitment rent structure are not related to intertemporal consumption smoothing, neither are they due to asymmetric information at renegotiation stage; rather they reflect the strategic value of pre-committed payment plan. These properties, however, are usually inconsistent with renegotiation-proofness. Several numerical examples are provided to illustrate these possibilities.

The main findings about long-term renegotiation-proof contracts are the following. First, it is shown that long-term renegotiation proof contracts are equivalent to efficient limited-commitment contracts with limited commitment placed solely on the part of the agent. As indicated earlier, the contracting technologies under the two regimes are quite different: With limited commitment the agent can unilaterally walk away at the beginning of each period; with long-term contracting and renegotiation, any abandonment or alteration of the contract must receive mutual consent. Moreover, under long-term renegotiation-proof contracts the agent in general receives a minimum level of lifetime utility over and above his reservation lifetime utility. This can occur even if in the static model the principal prefers to implement the minimum effort and to pay the agent the reservation utility.

Second, if both the principal and the agent are risk neutral toward income and if the agent's reservation utility is sufficiently high, then efficient full-commitment contracts and renegotiation-proof contracts coincide and both can be implemented by a sequence of one-period contracts. This result generalizes Fudenberg, Holmstrom and Milgrom [7]. However, if the agent's reservation utility is below a certain level, efficient long-term full-commitment contracts are not renegotiation proof. Third, if the agent is risk averse, long-term full-commitment contracts in general are not renegotiation proof.

I also offer a convergence result that links renegotiation-proof contracts in the finite-horizon setting to that in the infinite-horizon setting, which also serves as an algorithm for computing renegotiation-proof value functions. I stress that it is important in the current analysis to allow random wage contracts and to weaken the notion of renegotiation-proofness somewhat in order to establish the existence of renegotiation-proof contracts, the link between renegotiation-proof and limited-commitment contracts, and the link between finite and infinite horizon renegotiation-proof contracts. Computed examples are provided to illustrate this point.

The analysis in this paper can be contrasted with the literature on contracting with asymmetric information and renegotiation. It is important in here that at every renegotiation stage the principal and the agent have symmetric information about their preferences over subsequent contingent outcomes. If there are asymmetric information at renegotiation stage (for example if the agent knows more about his preferences than the principal), then adverse selection problem arises and the analysis will be quite different. Fudenberg and Tirole [8] and Park [14] analyze renegotiation with asymmetric information in principal-agent contracting. Dewatripont [5] and Laffont and Tirole [12] analyze long-term renegotiation-proof contracts, while Hart and Tirole [10] and Rey and Salanie [19] analyze links between long-term renegotiation-proof contracts and limited commitment short-term contracts. Moreover, the complete contracting approach of this paper also differs from the literature on incomplete contracting with renegotiation.

The rest of the paper is organized as follows. Section 2 provides a simple example to motivate the idea. Section 3 spells out the details of the model. Section 4 introduces the concept of renegotiation-proofness. Section 5 presents the existence and some characterization results. Section 6 provides further characterizations and several computed examples. Section 7 studies the link between finite-horizon and infinite-horizon RP contracts. Section 8 deals with the two-action two-outcome case where stronger results are possible. Section 9 concludes.

2 An Example

The following example demonstrates that efficient long-term contracts may not be renegotiation-proof. Consider a two-period repeated principal-agent model as follows. The agent has two hidden actions a_2, a_1, which also represent the utility costs of exerting the actions. Without loss of generality, assume $a_2 > a_1 = 0$. There are two possible output levels $y_2 > y_1 = 0$. To focus on the incentive structure, I assume both the agent and the principal are risk neutral toward consumption, so intertemporal smoothing is irrelevant. The agent has a period utility function given by $c - a$ and the principal cares only about her expected net revenue. Let p_2 be the probability that outcome y_2 will occur when the agent chooses effort a_2, and p_1 be the probability that outcome y_2 will occur when the agent chooses effort a_1. The agent's reservation utility level is normalized to zero. The agent's consumption level must be non-negative at all times.

Consider the following set of parameters: $p_2 = 0.8, p_1 = 0.2, y_2 = 1, e_2 = 0.3$. One can verify that $p_2y_2 - a_2 > p_1y_2 - a_1$, which says it is jointly efficient to exert high effort a_2.

Start with the one-period model. To implement effort choice a_2, the principal promises a wage w_2 if output y_2 occurs and a wage $w_1 = 0$ if output y_1 occurs. The agent's incentive compatibility condition requires

$$p_2w_2 - a_2 \geq p_1w_2 - a_1$$

which implies that the minimum wage w_2 the principal must pay to induce effort a_2 is given by

$$w_2 = \frac{a_2 - a_1}{p_2 - p_1} = \frac{0.3}{0.8 - 0.2} = 0.5.$$

For $i = 1, 2$, let R_i and r_i be the expected utility of the principal and the agent respectively if effort a_i is implemented with minimum cost to the principal. One can derive that $r_1 = 0$, $R_1 = p_1 y_2 = 0.2$, $r_2 = p_2 w_2 - a_2 = 0.8 \times 0.5 - 0.3 = 0.1$, and $R_2 = p_2(y_2 - w_2) = 0.4$. Clearly in the one-period model, the principal prefers to implement effort a_2 although she must pay the agent positive rent.

Now consider the two-period model. What is the optimal long-term contract for the principal? Given that there is no need for intertemporal smoothing, the natural starting point is the repetition of the one-period optimal contract ($w_2 = 0.5, w_1 = 0$). The principal's expected utility would be $(1+\delta)R_2$ and the agent's expected utility would be $(1+\delta)r_2$, where $0 < \delta \leq 1$ is the common discount factor. Can the principal do better than this? The answer is YES.

Consider the following two-period contract: if first period outcome is y_2, the agent gets wage payment $\hat{w}$ and is promised the wage contract ($w_2 = 0.5, w_1 = 0$) for the next period; if first period outcome is y_1, the agent gets zero wage payment and is promised wage contract ($w_2' = 0, w_1 = 0$) in the next period. The wage payment $\hat{w}$ is chosen to satisfy

$$p_2(\hat{w} + \delta r_2) - e_2 = p_1(\hat{w} + \delta r_2) - a_1$$

which ensures that effort a_2 will be implemented in period 1. It follows that

$$\hat{w} = \frac{a_2}{p_2 - p_1} - \delta r_2 = w_2 - \delta r_2 = 0.5 - 0.1\delta \tag{1}$$

Given the period 2 contingent wage contracts, the agent will choose a_1 in period 2 if period 1 outcome was y_1 and will choose a_2 in period 2 if period 1 outcome was y_2. By equation (1), the principal's ex ante expected utility is given as

$$p_2(y_2 - \hat{w} + \delta R_2) + (1 - p_2)\delta R_1$$

$$= p_2(y_2 - w_2 + \delta r_2 + \delta R_2) + (1 - p_2)\delta R_1$$

$$= p_2(y_2 - w_2) + \delta p_2 r_2 + \delta p_2 R_2 + (1 - p_2)\delta R_1$$

$$= (1+\delta)R_2 + \delta p_2 r_2 - (1 - p_2)\delta(R_2 - R_1).$$

It follows that the principal prefers this contract if the following condition is satisfied

$$p_2 r_2 - (1 - p_2)(R_2 - R_1) > 0,$$

which is guaranteed by the given parameters: $p_2 r_2 - (1 - p_2)(R_2 - R_1) = 0.8 \times 0.1 - 0.2 \times (0.4 - 0.2) = 0.04 > 0$.

The above long-term contract involves the use of ex post inefficient contract in period 2 when first-period output is y_1, so it will be subject to renegotiation if such opportunity exists.

3 The Model

3.1 The Stage Model

The model is a repeated version of the standard principal-agent model. Time is discrete: $t = 1, 2, ...$ In each period, a wage scheme goes into effect. The agent then takes a hidden action a from a finite set A. Each action $a \in A$ induces a probability distribution over a finite set Y of publicly observable outputs; in particular, for every action $a \in A$, $p(y|a)$ is the probability that output $y \in Y$ occurs. If the realized output is y, then the agent gets paid $w(y)$ according to current wage scheme $w(\cdot)$. The agent's utility function is given by $u(w) - g(a)$, where $w \in \Re_+$ is wage payment and $a \in A$ is action. The principal's utility is $y - w$.

In each period, the wage scheme can be randomly drawn from a menu of wage schemes according to some probability distribution. If the menu contains J wage schemes: $\{w^1(\cdot), ..., w^J(\cdot)\}$, each scheme $w^j(\cdot)$ is drawn with probability π^j, and given each wage scheme $w^j(\cdot)$ the agent's action is a^j, then the principal's expected payoff is

$$\sum_{j=1}^{J} \sum_{y \in Y} \pi^j p(y|a^j)(y - w^j(y))$$

and the agent's expected payoff is

$$\sum_{j=1}^{J} \sum_{y \in Y} \pi^j p(y|a^j)(u(w^j(y)) - g(a^j)).$$

3.2 The Repeated Model

In the repeated model, the public history at the beginning of period t is $h^t = ((w_1, y_1), ..., (w_{t-1}, y_{t-1}))$, where each pair (w_τ, y_τ) records the realized wage scheme and the realized output in period τ. Let $H^t = (\Re_+^{|Y|} \times Y)^{t-1}$ be the set of all possible histories at the beginning of period t. [4]

A *contract* σ consists of a pair of plans: a contingent wage plan $\underset{\sim}{w}$ and a contingent action plan $\underset{\sim}{s}$ recommended for the agent. A *wage plan* $\underset{\sim}{w}$ is a sequence of maps $\{\mathbf{w}_t\}_{t=1}^{\infty}$ from time-t public history to the set of random wage schemes; each $\mathbf{w}_t(h^t)$ specifies a probability distribution on $\Re_+^{|Y|}$, the set of deterministic wage schemes. A realized wage scheme $w_t \in \Re_+^{|Y|}$ specifies wage payment $w_t(y)$ for each output realization $y_t = y \in Y$. A recommended *action plan* $\underset{\sim}{s}$ for the agent is also a sequence of maps $\{s_t\}_{t=1}^{\infty}$ from time-t histories and time-t wage schemes to set A; each $s_t(h^t, w_t)$ specifies the action of the agent in period t given realized time-t history h^t and realized time-t wage scheme w_t.[5] Finally, let $\mathcal{W}$ be the space of all possible wage plans, $\mathcal{S}$ be the space of all possible action plans, and Σ be the space of all possible contracts.

Both the principal and the agent maximize the sum of expected discounted period payoffs, using a common discount factor $\delta \in (0, 1)$. Specifically, the agent's expected sum of discounted period utility is given by a function $v_1 : \mathcal{W} \times \mathcal{S} \longrightarrow \Re$ defined as follows:

$$v_1(\underset{\sim}{w}, \underset{\sim}{s}) = E \sum_{t=1}^{\infty} \delta^{t-1} [u(w_t) - g(s_t)],$$

and the principal's expected sum of discounted period utility is given by a function $v_0 : \mathcal{W} \times \mathcal{S} \longrightarrow \Re$ defined as follows:

$$v_0(\underset{\sim}{w}, \underset{\sim}{s}) = E \sum_{t=1}^{\infty} \delta^{t-1} [y_t - w_t],$$

where the two expectations are taken with respect to the distribution over histories that is generated by action plan $\underset{\sim}{s}$ and wage plan $\underset{\sim}{w}$. $v_0(\sigma)$ and $v_1(\sigma)$ are also referred to as the *value* of the contract to the principal and the value to the agent, respectively.

[4] Note that H^1 is the null set.

[5] The action plans specify only pure actions; this is without loss of generality: in case there are multiple optimal actions for the agent the contract picks the most desirable one for the principal.

It is useful to introduce the notion of *continuation*. Given a contract $\sigma = (\underset{\sim}{w}, \underset{\sim}{s})$. Fix an history h^t, for $t = 1, 2, ...$ The continuation action plan $\underset{\sim}{s}|h^t$ is the restriction of $\underset{\sim}{s}$ to histories h^τ for $\tau \geq t$ whose first t period components coincide with h^t. Continuation wage plan $\underset{\sim}{w}|h^t$ is defined analogously. Continuation contract $\sigma|h^t$ is the pair $(\underset{\sim}{w}|h^t, \underset{\sim}{s}|h^t)$.

A contract $\sigma = (\underset{\sim}{w}, \underset{\sim}{s})$ is **incentive compatible** if given wage plan $\underset{\sim}{w}$, action plan $\underset{\sim}{s}$ is optimal for the agent: $v_1(\underset{\sim}{w}, \underset{\sim}{s}) \geq v_1(\underset{\sim}{w}, \underset{\sim}{s}')$, for all $\underset{\sim}{s}' \in \mathcal{S}$.

Given some constants $\underline{\xi}$ and $\bar{\xi}$, a contract σ is **feasible** if $v_1(\sigma|h^t) \in [\underline{\xi}, \bar{\xi}]$, for every history h^t. The upper bound on the agent's lifetime utility reflects the limited ability of the principal in making wage payments: although there is no explicit limit on wage payment, the lifetime utility of the agent can not exceed $\bar{\xi}$, which implicitly imposes an upper bound on wage payments in each period. The lower bound $\underline{\xi}$ reflects limited liability of the agent. For example, if wage can not be negative and if the agent can choose an action that costs him nothing (as will be assumed below), then the agent's lifetime utility can not be below zero: $\underline{\xi} = 0$.

Finally, the following assumptions are made throughout the paper:

A 1. The agent has a separable utility function: $u(c) - g(a)$, with $c \geq 0$, $u'(\cdot) > 0$, $u''(\cdot) \leq 0$, $g(a) \geq 0$ for all $a \in A$, and there is an $\underline{a} \in A$ such that $g(\underline{a}) = 0$.

A 2. (Full Support) For all $a \in A$ and for all $y \in Y$, $p(y|a) > 0$.

Note that under the full support assumption, a contract σ is incentive compatible if and only if every continuation contract $\sigma|h^t$ is incentive compatible.

4 Renegotiation-Proofness

This section introduces the concept of renegotiation-proofness. The idea is not to spell out the detailed process of renegotiation, but rather to assume that the final effective contract leaves no room for further renegotiation that can lead to welfare improvement for some party without hurting others, i.e. Pareto improvement.

4.1 The Preliminary Axioms

If a contract is to be renegotiated only once, then it is clear that renegotiation proof contract is equivalent to Pareto optimal contract. In a multi-period model where renegotiation can take place in every period, renegotiation proof contract should be Pareto optimal subject to the constraint that the principal and the agent can not achieve Pareto improvement through renegotiation in all future dates. If the number of periods is finite, this constrained optimality can be defined recursively in every period using backward induction. In the current infinite-horizon model where renegotiation can take place infinitely many times, it is impossible to apply backward induction. The method, however, does lend its recursive nature to the development of a new concept: renegotiation proof contracts should be Pareto optimal subject to the constraint that any continuation contract satisfies the same requirement. The rest of this section is devoted to formalizing this seemingly cyclical idea.

The recursive definition of renegotiation-proofness for the finite-horizon setting motivates the following two conditions that one would like a renegotiation proof contract in the infinite-horizon setting to satisfy.

Axiom 1. (Recursion) A contract is renegotiation proof if and only if every continuation contract is renegotiation proof.

Axiom 2. (Pareto Optimality) A renegotiation proof contract is not Pareto dominated by any other renegotiation proof contract.

The goal is to characterize contracts that satisfy these two axioms, and then use the result to derive an *operational* definition of renegotiation-proofness. To this end, next I consider the set of values that contracts can deliver to the principal and the agent.

4.2 Value Functions and Renegotiation-Proofness

A *value function* f maps each promised payoff ξ' of the agent to the optimal payoff $f(\xi')$ of the principal, achievable using incentive compatible and feasible contracts.

Suppose $f(\cdot)$ is an optimal *continuation* value function, i.e. if the agent is promised an expected payoff ξ' from next period onwards then the principal's optimal continuation value is $f(\xi')$. Consider how one can find the principal's optimal value function for today. The idea is to design current wage scheme and promised utility scheme to *generate* the principal's optimal value for today, given optimal continuation value function $f(\cdot)$. Formally, one has the following notion of generating.

DEFINITION 1. Let f be a real-valued continuous function defined on an interval $[\ell, \bar{\xi}] \subset \Re_+$. A function $\Gamma f : [\delta\ell, \bar{\xi}] \longrightarrow \Re$ is *generated* by f if for every $\xi \in [\delta\ell, \bar{\xi}]$, $\Gamma f(\xi)$ is given by:

$$\Gamma f(\xi) = \max_{(\pi^j, w^j, \xi^j, a^j)} \sum_{j \in J} \pi^j \sum_{y \in Y} p(y|a^j) \Big[y - w^j(y) + \delta f(\xi^j(y)) \Big] \tag{2}$$

s.t.

$$\sum_{j \in J} \pi^j \Big[\sum_{y \in Y} p(y|a^j) \big(u(w^j(y)) + \delta \xi^j(y) \big) - g(a^j) \Big] = \xi, \tag{3}$$

for all $j \in J$ and for all $a \in A$,

$$\sum_{y \in Y} [p(y|a^j) - p(y|a)] \big(u(w^j(y)) + \delta \xi^j(y) \big) \geq g(a^j) - g(a), \tag{4}$$

where for all $j \in J,\ w^j : Y \longrightarrow R_+,\ \xi^j : Y \longrightarrow [\ell, \bar{\xi}],\ a^j \in A,\ \pi^j \geq 0,$ and $\sum_{j \in J} \pi^j = 1$. For the record, Γ is called the *generating operator*.

Note that random wage scheme is permitted: the choice variables are a distribution (π^j) over a menu $(w^j(\cdot), \xi^j(\cdot), a^j)_{j \in J}$, where for each $j \in J$, $\xi^j(y)$ is the promised payoff to the agent given current output y. The value function $\Gamma f(\cdot)$ is the upper frontier of the convex hull of the value function without randomization. It follows that $\Gamma f(\cdot)$ is concave.

Fix a real-valued continuous function f defined on a positive interval $[\ell, \bar{\xi}]$. Define Φ as the *Pareto generating operator*, which is the composition of the generating operator Γ and the operation of taking Pareto frontier, so $\Phi(f)$ is the Pareto frontier of the generated value function Γf.

DEFINITION 2. A real-valued function f defined on a positive interval $[\ell, \bar{\xi}]$, is *Self-Pareto-Generating* if the Pareto frontier of the generated value function Γf is identical to f, i.e. if $\Phi(f) = f$. [6]

[6] Ray [17] calls such set internally renegotiation proof. Bergin and MacLeod [3] also introduce a similar concept, called full recursive efficiency.

Proposition 1. *Let f be the value function the graph of which is the set of all value pairs delivered by renegotiation-proof contracts that satisfy Axioms 1 and 2. Then $\Phi(f) = f$.*

Proof. Any point on the Pareto frontier $\Phi(f)$ corresponds to a renegotiation-proof contract: it will not be renegotiated today, and its continuation contracts are all renegotiation-proof. Thus $\text{Graph}(\Phi(f)) \subseteq \text{Graph}(f)$. On the other hand, a point on $\text{Graph}(f)$ can be generated by value function f, because the continuation contracts of a renegotiation proof contract should also be renegotiation proof. It follows that f must be part of the Pareto frontier $\Phi(f)$: $\text{Graph}(f) \subseteq \text{Graph}(\Phi(f))$. Hence, $\Phi(f) = f$. Q.E.D.

Proposition 1 provides the basis for a definition of renegotiation-proofness that satisfies Axioms 1 and 2. But we should note that Self-Pareto-Generating is only a necessary condition following the two axioms; we still need to keep Pareto Optimality (Axiom 2). In light of this, two concepts are introduced below.

Definition 3. A contract σ is *(weak) renegotiation proof* if the set of value pairs delivered by all of its continuation contracts is a subset of the graph of a Self-Pareto-Generating (henceforth also called renegotiation proof) value function.

Definition 4. A contract σ is *strong renegotiation proof* if it is weak renegotiation proof and it is not Pareto dominated by any weak renegotiation proof contracts. [7]

Note that the value function of strong renegotiation proof contracts, if exists, must be unique. In the rest of the paper, unless otherwise indicated, I will use renegotiation proof to mean *weak* renegotiation proof, and use renegotiation proof and Self-Pareto-Generating interchangeably.

Figure 1 illustrates a renegotiation proof value function, the graph of which is the curve BC.

[Insert Figure 1 here]

[7] This pair of concepts may be contrasted with the concepts (with the same namesakes) introduced by Farrell and Maskin [6] and those by Bernheim and Ray [4]. The weak renegotiation-proofness here is stronger than theirs.

A special case where efficient full-commitment contracts are in fact renegotiation-proof is when the principal can punish the agent arbitrarily severely in a single period, as shown in the following proposition. Similar result was also obtained by Fudenberg, Holmstrom and Milgrom [7]. It is included here for completeness.

Proposition 2. *Assume A1-A2 and* $u(0) = -\infty$. *Then every continuation contract of an efficient full-commitment contract is efficient.*

Note that this "recursive efficiency" property clearly implies renegotiation-proofness. The proof of the proposition is sketched as follows. If any continuation contract is inefficient, then replace it by a Pareto superior incentive compatible and feasible continuation contract. If the agent's continuation payoff is not increased as a result, then the agent's incentives are not affected and the principal's continuation payoff must be increased. The resulted new contract then Pareto dominates the old one, a contradiction. If the agent's continuation payoff goes up after the replacement, then reduce the agent's wage payments in the previous period by some amount independent of output realizations so as to bring her expected payoff at that history back to the original level promised by the old contract. The agent's incentives are unaffected after that and the principal's expected payoff is increased, which again leads to a contradiction.

For the rest of this paper, I will assume that $u(0) = 0$ and the agent's reservation utility is zero. As seen from the example in the previous section, ex ante efficient contracts in general are not renegotiation proof.

For later reference, I record the following two results concerning generating operator Γ, the proof of the first lemma is obvious and omitted.

Lemma 1. *Let* $f_1 : [\ell_1, \bar{\xi}] \longrightarrow \Re$, *and* $f_2 : [\ell_2, \bar{\xi}] \longrightarrow \Re$ *be two continuous functions. Suppose* $\ell_2 \geq \ell_1$, *and* $f_1(\xi) \geq f_2(\xi)$, *for all* $\xi \in [\ell_2, \bar{\xi}]$. *Then* $\Gamma f_1(\xi) \geq \Gamma f_2(\xi)$ *whenever both sides are defined.*

Lemma 2. *There exist some* $\bar{w} > 0$ *and some* $M > 0$ *such that given any continuous real-valued function* f *defined on a positive interval* $[\ell, \bar{\xi}]$, *(a) the generated function* $\Gamma(f)$ *is concave, and the left and right derivatives of* $\Gamma(f)$ *are bounded from below by* $-M$; *(b) wage payments are bounded from above by* $\bar{w}$.

Proof. To prove part (b), let $\underline{p}$ be the $\min\{p(y|a) : \forall a \in A,\ \forall y \in Y.\}$; let $\bar{g} = \max\{g(a) : \forall a \in A\}$; and let $\bar{w}$ be such that $u(\bar{w})\underline{p} - \bar{g} = \bar{\xi}$. Concavity of $\Gamma(f)$

is already established. To prove the rest of part (a), by inspecting Eq. (2), and by a variation argument, one finds that the left derivative of $\Gamma(f)$ at $\bar{\xi}$ will be no smaller than

$$\sum_j \pi^j \sum_{y \in Y} -\frac{p(y|a^j)}{u'(w^j(y))}$$

which together with part (b) implies that the left derivative of $\Gamma(f)$ at $\bar{\xi}$ is no smaller than $-M \equiv -\frac{1}{u'(\bar{w})}$. Since $\Gamma(f)$ is concave, the left and right derivatives of $\Gamma(f)$ at every point (derivatives of support functions) must be bigger than or equal to $-M$. Q.E.D.

The concept of RP can be demanding when it comes to existence. A weaker concept, *principal renegotiation proof*, will guarantee existence.

DEFINITION 5. Let $f : [\ell, \bar{\xi}] \longrightarrow R$ be a nonincreasing function. Then f is said to be *principal renegotiation proof* (PRP) if $\Gamma f(\xi) = f(\xi)$, for all $\xi \in [\ell, \bar{\xi}]$ and $f(\ell) \geq \Gamma f(\xi)$, for all ξ in the domain of Γf.[8]

To paraphrase, f can not contain any point that makes the principal *strictly* better off without hurting the agent, compared to other points on f, but f may contain points that make the agent strictly better off without hurting the principal, compared to other points on f.[9] The concept thus implicitly assumes that it is the principal who initiates renegotiation and she will do so only if there is strict gain for herself. Apparently, an RP set is also a PRP set, but the reverse is not necessarily true.

5 Existence and Characterization

In this section, I show that the value function of renegotiation proof contracts has a simple characterization: it is the value function of optimal contracts with an appropriate lower bound placed on the agent's promised utility. Thus, RP contracts are equivalent to optimal contracts with limited commitment, where the agent faces an appropriate outside option in each date and can choose to walk away forever.

[8]When $\ell = \bar{\xi}$, function f is defined on the singleton set $\{\bar{\xi}\}$ and is trivially nonincreasing.

[9]This is somewhere between weak Pareto optimality and full Pareto optimality: the principal and the agent are treated asymmetrically.

5.1 Optimal Contracting with Limited Commitment

Consider a class of optimal contracting problems where the agent has limited commitment so his promised utility is bounded from below by some value $\ell \in [0, \bar{\xi}]$ of an outside option.

Given a promised utility $\xi \in [\ell, \bar{\xi}]$ to the agent,[10] let $V(\xi, \ell)$ be the maximum value the principal can obtain using incentive compatible and feasible contracts with the additional constraint that every continuation contract promises the agent a payoff within $[\ell, \bar{\xi}]$.

Following standard argument,[11] the optimal value function $V(\cdot, \ell) : [\ell, \bar{\xi}] \longrightarrow \Re$ satisfies the following functional equation:

$$V(\xi, \ell) = \max_{(\pi^j, \mathbf{w}^j, \xi^j, a^j)} \sum_{j \in J} \pi^j \sum_{y \in Y} p(y|a^j) \Big[y - w^j(y) + \delta V(\xi^j(y), \ell) \Big] \tag{5}$$

s.t.

$$\sum_{j \in J} \pi_j \Big[\sum_{y \in Y} p(y|a^j) \big(u(w^j(y)) + \delta \xi^j(y) \big) - g(a^j) \Big] = \xi$$

$$\sum_{y \in Y} [p(y|a^j) - p(y|a)] \big(u(w^j(y)) + \delta \xi^j(y) \big) \geq g(a^j) - g(a), \ \ \forall a \in A, \ \forall j \in J,$$

where for all $j \in J, \ w^j : Y \longrightarrow R_+, \ \xi^j : Y \longrightarrow [\ell, \bar{\xi}], \ a^j \in A, \ \pi^j \geq 0$, and $\sum_{j \in J} \pi^j = 1$.

Again, random wage scheme is permitted: the choice variables are a distribution (π^j) over a menu $(w^j(\cdot), \xi^j(\cdot), a^j)_{j \in J}$. The value function $V(\cdot, \ell)$ is the upper frontier of the convex hull of the value function without randomization. It follows that $V(\cdot, \ell)$ is concave.

Let T_ℓ be the contraction mapping operator embedded in functional equation (5). Given a real-valued function defined on an positive interval $[\ell, \bar{\xi}]$, function $T_\ell f(\cdot, \ell)$ is the restriction of the generated function $\Gamma f(\cdot, \ell)$ to $[\ell, \bar{\xi}]$: for all $\xi \in [\ell, \bar{\xi}]$, $\Gamma f(\xi, \ell) = T_\ell f(\xi, \ell)$, but the domain of $\Gamma f(\cdot, \ell)$ is $[\delta \ell, \bar{\xi}]$.

The following result compares the effects of different values of the agent's outside option on the principal's welfare: higher value of the outside option reduces the principal's payoff.

[10] Note that the promised utility ξ is not the standard reservation utility of the agent because the principal is constrained to deliver to the agent an expected payoff *exactly equal* to ξ.

[11] For instance, see Green ([9]) and Spear and Srivastava ([22]).

Lemma 3. *If* $0 \leq \ell' < \ell \leq \bar{\xi}$*, then* $V(\xi, \ell') \geq V(\xi, \ell)$ *for all* $\xi \in [\ell, \bar{\xi}]$*; and* $\Gamma V(\xi, \ell') \geq \Gamma V(\xi, \ell)$ *whenever both sides are defined.*

Proof of Lemma 3. Let $f : [\ell', \bar{\xi}] \longrightarrow \Re$ and $h : [\ell, \bar{\xi}] \longrightarrow \Re$ be identically zero on their respective domains. Then by Lemma 1, $\Gamma f(\xi) \geq \Gamma h(\xi)$, for all $\xi \in [\ell, \bar{\xi}]$. Hence $T_{\ell'} f(\xi) \geq T_\ell h(\xi)$, for all $\xi \in [\ell, \bar{\xi}]$. Again, by Lemma 1, $\Gamma T_{\ell'} f(\xi) \geq \Gamma T_\ell h(\xi)$, for all $\xi \in [\ell, \bar{\xi}]$. Hence $T_{\ell'}^2 f(\xi) \geq T_\ell^2 h(\xi)$, for all $\xi \in [\ell, \bar{\xi}]$. Keep applying Lemma 1, one has for all $n = 1, 2, ...$ and for all $\xi \in [\ell, \bar{\xi}]$, $T_{\ell'}^n f(\xi) \geq T_\ell^n h(\xi)$. Note that The two sequences converge to $V(\cdot, \ell')$ and $V(\cdot, \ell)$ respectively. The first statement follows, and so does the second using Lemma 1 one more time. Q.E.D.

The next important result shows that the family of value functions $V(\cdot, \cdot)$ has a certain sense of continuity with respect to its second argument ℓ: as the lower bounds get closer, the value functions also get closer *uniformly* on the intersection of their domains. Formally:

Lemma 4. *(Continuity) For all* $\ell \in [0, \bar{\xi}]$*, it holds that*

$$\lim_{\ell' \to \ell} \left\{ \sup_{\xi \in [\max(\ell, \ell'), \bar{\xi}]} |V(\xi, \ell) - V(\xi, \ell')| \right\} = 0$$

Proof of Lemma 4. See the Appendix. Q.E.D.

5.2 Existence and Characterization

The existence of Principal-Renegotiation-Proof value function and the main characterization result about the RP and PRP value functions will be established.

Let $L = \{\ell \in [0, \bar{\xi}] : V(\cdot, \ell) \text{ is nonincreasing.}\}$. Note that L is nonempty: $\bar{\xi} \in L$. Let $\ell^* = \inf L$.

Lemma 5. *Function* $V(\cdot, \ell^*)$ *is nonincreasing on* $[\ell^*, \bar{\xi}]$.

Proof. The result trivially holds if $\ell^* = \bar{\xi}$. Suppose $\ell^* < \bar{\xi}$. Let $\langle \ell_n \rangle$ be a sequence in L which converges to ℓ^*. Lemma 4 ensures that $\langle V(\cdot, \ell_n) \rangle$ converges in metric d to $V(\cdot, \ell^*)$. Since all $\langle V(\cdot, \ell_n) \rangle$ are nonincreasing, it follows $V(\cdot, \ell^*)$ is also nonincreasing. Q.E.D.

The following characterization result also establishes the existence of PRP value function.

Proposition 3. *Value function $V(\cdot, \ell^*)$ is PRP. If $V(\cdot, \ell^*)$ is strictly decreasing, then it is RP, and in fact strong renegotiation proof.*

Proof. The proof is by contradiction. Suppose $V(\cdot, \ell^*)$ is not PRP (or not RP if it is decreasing). Then there must exist some $\hat{\xi} \in [0, \ell^*)$ such that $\Gamma V(\hat{\xi}, \ell^*) > V(\ell^*, \ell^*)$. I will show that then there exists some $\ell' < \ell^*$ such that $\Gamma V(\hat{\xi}, \ell') < \Gamma V(\hat{\xi}, \ell^*)$, which is a contradiction by Lemma 3. For later reference, let $\Delta \equiv \Gamma V(\hat{\xi}, \ell^*) - V(\ell^*, \ell^*) > 0$.

By Lemma 4, given any $\epsilon \in (0, \frac{\Delta}{2})$, there exists some $\eta' > 0$ such that for all $\ell' \in (\ell^* - \eta', \ell^*)$,

$$|V(\ell^*, \ell^*) - V(\ell^*, \ell')| < \epsilon.$$

Since $V(\ell^*, \ell^*) \leq V(\ell^*, \ell')$ by Lemma 3, it follows that

$$V(\ell^*, \ell^*) + \epsilon > V(\ell^*, \ell'). \tag{6}$$

Now by Lemma 2, there exists some constant $M > 0$ such that $\forall \ell' \in [0, \bar{\xi}]$ and $\forall \xi \in [\ell', \ell^*]$,

$$V(\ell^*, \ell') + (\ell^* - \xi)M \geq V(\xi, \ell'). \tag{7}$$

Let $\eta = \min\{\eta', \frac{\Delta}{2M}\}$. Then for $\ell' \in (\ell^* - \eta, \ell^*)$ and $\forall \xi \in [\ell', \bar{\xi}]$,

$$\eta M \geq (\ell^* - \xi)M. \tag{8}$$

Adding up Eq.(8) and Eq. (6), one has

$$V(\ell^*, \ell^*) + \epsilon + \eta M \geq V(\ell^*, \ell') + (\ell^* - \xi)M. \tag{9}$$

Eq.(9) and Eq. (7) imply that $\forall \ell' \in (\ell^* - \eta, \ell^*)$ and $\forall \xi \in [\ell', \bar{\xi}]$,

$$V(\ell^*, \ell^*) + \epsilon + \eta M \geq V(\xi, \ell'),$$

which further implies,

$$V(\ell^*, \ell^*) + \Delta = V(\ell^*, \ell^*) + \frac{\Delta}{2} + \frac{\Delta}{2M}M > V(\xi, \ell'). \tag{10}$$

By the definition of ℓ^*, for any $\ell' \in [0, \ell^*)$, function $V(\cdot, \ell')$ attains its maximum value somewhere in $(\ell', \bar{\xi}]$. Concavity of the generated function $\Gamma V(\cdot, \ell')$ and the fact that $V(\cdot, \ell')$ and $\Gamma V(\cdot, \ell')$ coincide on $[\ell', \bar{\xi}]$ imply that $\Gamma V(\cdot, \ell')$ also attains its (identical) maximum value at the same points.

Thus by Eq.(10) the maximum of $V(\cdot, \ell')$, hence the *maximum* of $\Gamma V(\cdot, \ell')$ will be less than $V(\ell^*, \ell^*) + \Delta = \Gamma V(\hat{\xi}, \ell^*)$. Therefore, $\Gamma V(\hat{\xi}, \ell') < \Gamma V(\hat{\xi}, \ell^*)$, which is impossible by Lemma 3, because $\ell' < \ell^*$.

To prove that a strictly decreasing $V(\cdot, \ell^*)$ is strong renegotiation proof, note that any weak renegotiation proof value function $V(\cdot, \ell)$ satisfies $TV(\cdot, \ell) = V(\cdot, \ell)$, so $\ell \in L$. Since $\ell^* = \inf L$, by Lemma 3, $V(\cdot, \ell^*)$ Pareto dominates any such $V(\cdot, \ell)$. *Q.E.D.*

[Insert Figure 2 here]

The logic of the proof is illustrated by Figure 2. The AD curve is for function $\Gamma V(\cdot, \ell^*)$ and the BC curve is for $\Gamma V(\cdot, \ell')$. Function $\Gamma V(\cdot, \ell^*)$ attains its maximum at $\hat{\xi}$. For all $\ell' < \ell^*$, $\Gamma V(\cdot, \ell')$ attains its maximum on $(\ell', \bar{\xi}]$. As $\ell' \to \ell^*$, the maximum of $\Gamma V(\cdot, \ell') \to \Gamma V(\cdot, \ell^*)$, which implies $\Gamma V(\hat{\xi}, \ell') < \Gamma V(\hat{\xi}, \ell^*)$, a contradiction.

This result establishes a connection between RP contracts and limited commitment contracts. It also validates an informal treatment of renegotiation-proof contracts in the dynamic contracting literature. In dynamic principal-agent contracting the optimal value of the principal as a function of the agent's promised utility in general is not a nonincreasing function. Thus a point on the upward-sloping portion of the value function may be subject to renegotiation. Downward-sloping value functions may be obtained if suitable lower bounds are placed on the agent's promised utility. It seems that the downward-sloping value function generated by the smallest such lower bound corresponds to some "renegotiation-proof" contracts.[12] Proposition 3 formally validates this method. However, one should note that this result is obtained by permitting public randomization and by weakening renegotiation-proofness to principal-renegotiation-proofness. The weaker concept indeed is crucial for existence, for Self-Pareto-Generating or renegotiation-proof value function may not exist in some cases (then the informal treatment would be questionable). See Example 5 in the next section for an illustration.

[12] For instance, Phelan and Townsend [16] hinted on such a treatment; Vincenzo Quadrini (2001) uses this method to study optimal renegotiation-proof financial contracts between an entrepreneur and an investor.

6 Further Characterization

In this section, I offer some characterizations of the renegotiation-proof value function and contrast the results with the full-commitment value function. One basic conclusion is that renegotiation-proofness imposes a lower bound ℓ^* on the agent's lifetime utility, which in general is above zero (the reservation level of utility) and below $\bar{\xi}$ (the maximum level of the agent's lifetime utility). I also characterize the relationship between RP value function and the Pareto frontier of the full-commitment value function.

6.1 Renegotiation raises agent's minimum payoff

Consider first the stage model. For each action $a \in A$, let $R(a)$ be the principal's maximum payoff by implementing a and let $r(a)$ be the agent's payoff thereof. If a can not be implemented, then $R(a) = -\infty$ and $r(a)$ is not defined. Clearly, $r(\underline{a}) = 0$. To implement any action a with utility cost $g(a) > 0$, it is necessary for the wage schedule $(w(y))_{y \in Y}$ to satisfy

$$r(a) = E[w(y)|a] - g(a) \geq E[w(y)|\underline{a}]$$

where $E[\cdot|a]$ is the expectation operator with respect to the probability distribution $p(\cdot|a)$ on Y. By the full support assumption A2, it follows that $r(a) > 0$.

PROPOSITION 4. *Suppose that for the one-period problem, the principal's maximum payoff $R(a^*)$ by implementing an action a^* with $g(a^*) > 0$ is strictly higher than the payoff $R(\underline{a})$ by implementing the least-cost-action $\underline{a}$. Then the lower bound $\ell^* > 0$.*

Proof. Suppose $\ell^* = 0$. Then the full-commitment value function $V^f(\cdot)$ and RP value function coincide, so $V^f(\cdot)$ is nonincreasing. If the agent is promised zero payoff, then the principal's maximum payoff is given by

$$R(\underline{a}) + \delta V^f(0).$$

The principal can always implement a^* and get a payoff at least as high as

$$R(a^*) + \delta V^f(0)$$

while providing the agent with payoff $r(a^*) > 0$. This contradicts that $V^f(\cdot)$ is nonincreasing. Q.E.D.

Wang (2000) obtains a similar result in the context of a finitely repeated principal-agent problem.

It is also of interest to know whether the only principal-renegotiation-proof (PRP) function is a singleton, i.e. $\ell^* = \bar{\xi}$. The next result shows that for practically all interesting contracting problems, PRP functions are not singleton.

PROPOSITION 5. *Suppose for the one-period contracting problem, the Pareto frontier of the principal's value function contains more than one point. Then* $\ell^* < \bar{\xi}$.

Proof. By Proposition 3, function $V(\cdot, \ell^*)$ is PRP. But the singleton function $V(\bar{\xi}, \bar{\xi})$ is not Self-Pareto-Generating, so is not PRP. Q.E.D.

Proposition 4 states that if the value function for the one-period problem is not nonincreasing then the full-commitment value function for the repeated problem is not nonincreasing either. The reverse, however, is not true: even if the one-period problem has nonincreasing (or even decreasing) value function, the value function for the repeated problem may not be nonincreasing and the minimum utility ℓ^* for the agent can still be bigger than zero. This is illustrated by the following example.

EXAMPLE 2

Both the principal and the agent are risk-neutral toward income. The agent has two actions a_1, a_2, which also represent the utility costs. Specifically, $a_1 = 0$, $a_2 = 0.2 + \varepsilon$, where $\varepsilon \geq 0$. There are two output levels: $y_1 = 0$, $y_2 = 1$. The probabilities are given by $p_1 \equiv \text{Prob}(y_2|a_1) = 0.4$, $p_2 \equiv \text{Prob}(y_2|a_2) = 0.8$. Recall that $R(a)$ and $r(a)$ are the payoffs of the principal and the agent respectively when a is implemented by the principal with minimum cost. It is straightforward to show that $r(a_1) = 0, R(a_1) = p_1 y_2 = 0.4$, $r(a_2) = p_1 w^* = 0.2 + \varepsilon$, and $R(a_2) = p_2(y_2 - w^*) = 0.4 - 2\varepsilon$, where $w^* \equiv a_2/(p_2 - p_1) = 0.5 + 2.5\varepsilon$. The value function for the one-period problem is nonincreasing for $\varepsilon = 0$ and decreasing for $\varepsilon > 0$. Figure 3 shows the value function, assuming that the maximum utility of the agent is 0.4 and $\varepsilon = 0.02$.

[INSERT FIGURE 3 HERE]

The full-commitment value function $V^f(\cdot)$, however, is not nonincreasing. To see this, first note that $V^f(0) = R(a_1)/(1-\delta)$. Next consider the following contract that the principal can offer. For periods $t > 2$ offer zero wage regardless of future and past output realizations. For period 1 given output realizations y_1, y_2, offer current wage payments $w(y_1) = 0, w(y_2) = w^* - \delta r(a_2)$ and promised utility for period 2, $\xi(y_1) = 0$, $\xi(y_2) = r(a_2)$, where $w^* = a_2/(p_2 - p_1)$. It is clear that these offers will implement a_2 in period 1. In period 2, if promised utility to the agent is $\xi(y_1) = 0$ then the principal offers zero wage regardless of output realization and the principal's payoff is equal to $R(a_1)$; if promised utility is equal to $\xi(y_2) = r(a_2)$ then the principal should offer to implement a_2 and derive current payoff equal to $R(a_2)$. In summary, the principal's payoff for the first two periods is given by

$$p_2\{y_2 - w_2(y_2) + \delta R(a_2)\} + (1 - p_2)\delta R(a_1)$$

$$= R(a_2) + \delta p_2(R(a_2) + r(a_2)) + (1 - p_2)\delta R(a_1)$$

which is larger than $(1+\delta)R(a_1)$ for small ε. (Clearly, if $\varepsilon = 0$, then $R(a_2) = R(a_1)$ and the difference is equal to $\delta p_2 r(a_2) > 0$.) This shows that $(0, V^f(0))$ is not a peak point of $V^f(\cdot)$ if ε is not too large.

This example also demonstrates the useful role of "deferred payments": It is cheaper for the principal to promise future payments rather than making current wage payments in order to implement high effort today. This is because the slope of the continuation value function is flat, so large future payments only result in moderate decrease in profits for the principal. Note this phenomenon is not related to intertemporal smoothing, as the agents are risk neutral toward income.

6.2 RP value function versus efficient frontier of full-commitment value function

I now turn to the relationship between RP value function and the Pareto frontier of the full-commitment value function. We have seen that if the agent's utility from income is unbounded from below these two objects coincide, so renegotiation-proofness does not have much of a bite. But in general RP value function and the Pareto frontier of the full-commitment value function will differ.

If both the principal and the agent are risk neutral toward income, then there is a range of promised payoffs for the agent on which the renegotiation-proof value function and the full-commitment value function coincide. To illustrate this, once again let $R(a)$ and $r(a)$ represent the payoffs of the principal and the agent respectively for the one-period problem if action $a \in A$ is implemented with minimum cost to the principal. Let $\hat{a} \in \arg\max_a R(a) + r(a)$, namely action $\hat{a}$ maximizes total surplus. Then it follows from risk neutrality that $\hat{a} \in \arg\max_a E_a[y - g(a)]$, i.e. $\hat{a}$ maximizes expected output net of utility cost. Recall that $V(\cdot, \ell^*)$ is the principal-renegotiation-proof value function.

Proposition 6. *Assume A1, A2 and that both the principal and the agent are risk neutral toward income. Then* $V(\xi, \ell^*) = V^f(\xi)$, *for all* $\xi \in [\ell, \bar{\xi}]$, *where* $\ell \equiv r(\hat{a})/(1-\delta)$.

Proof. The conclusion will follow if we can show that $V(\xi, \ell) = V^f(\xi)$ for all $\xi \in [\ell, \bar{\xi}]$. Need only to show that $V(\xi, \ell) \geq V^f(\xi)$ for all $\xi \in [\ell, \bar{\xi}]$, because by Lemma 3, $V(\xi, \ell) \leq V^f(\xi)$ for all $\xi \in [\ell, \bar{\xi}]$.

Fix $\xi \in [\ell, \bar{\xi}]$. Let σ be a full-commitment contract that delivers payoffs $V^f(\xi)$ and ξ to the principal and the agent respectively. Since agents are risk neutral, it follows that

$$V^f(\xi) + \xi = E_\sigma[\sum_{t=1}^{\infty}(y_t - g(a_t))]$$

where the expectation is taken with respect to the distribution generated by σ over outputs and pure actions.

On the other hand, if in each period the principal offers the same static contract that implements action $\hat{a}$ and gives the agent per-period payoff $(1-\delta)\xi$ (which is possible because $(1-\delta)\xi \geq r(\hat{a})$) , then the principal obtains payoff V_p which is given by

$$V_p + \xi = (E_{\hat{a}}(y) - g(\hat{a}))/1 - \delta.$$

But

$$(E_{\hat{a}}(y) - g(\hat{a}))/1 - \delta \geq E_\sigma[\sum_{t=1}^{\infty}(y_t - g(a_t))] = V^f(\xi) + \xi.$$

Hence $V_p \geq V^f(\xi)$. It follows that $V(\xi, \ell) = V^f(\xi)$, for all $\xi \in [\ell, \bar{\xi}]$. Q.E.D.

Thus when the principal and the agent are both risk neutral toward income, there is a critical level of payoff for the agent above which the efficient full-commitment contracts are renegotiation-proof; furthermore, the payoff outcomes of the efficient full-commitment contracts can be attained using stationary one-period contracts. Fudenberg, Holmstrom, and Milgrom (1990) obtains a special case of this result. They show that if both agents are risk neutral and the principal's expected payoff is zero then efficient full-commitment contracts are renegotiation-proof and can be implemented by a sequence of short-term contracts. Note that when the principal's expected payoff is zero the payoff of the agent in general is above the critical level identified in here.

Although the renegotiation-proof value function may coincide with part of the full-commitment value function, in general the Pareto frontier of the full-commitment value function is "larger" than the renegotiation-proof value function. Namely, there are efficient full-commitment contracts that are not renegotiation proof, even if both agents are risk neutral. The following example illustrates this point.

Example 3

Consider a class of problems which includes Example 1 in Section 2. Both the principal and the agent are risk neutral; the agent has two actions $a_1 = 0, a_2 > 0$, which also identify the respective utility costs of taking the actions; there are two output levels y_1, y_2. Again let R_i and r_i represent the payoffs of the principal and the agent respectively if action a_i, $i = 1, 2$, is implemented with minimum cost to the principal. Assume that the stage model satisfies the following condition:

A 3. $R_2 > R_1, r_2 > r_1 = 0$. $p_2 r_2 - (1 - p_2)(R_2 - R_1) > 0$.

Lemma 6. *Assume A1-3 and that both the principal and the agent are risk neutral toward income. Let ℓ^f be the largest maximizer of the full-commitment value function $V^f(\cdot)$. Then (i) $\ell^f < \ell^*$; (ii) the renegotiation-proof value function $V(\cdot, \ell^*)$ coincide with $V^f(\cdot)$ on $[\ell^*, \bar{\xi}]$.*

Proof. By Lemma 10, $\ell^* = r_2/(1 - \delta)$. Part (ii) then follows from Proposition 6. It follows from Assumption A3 and the logic of Example 1 that the principal can obtain payoff higher than $V(\ell^*, \ell^*)$ by offering the agent some payoff below ℓ^*, which proves part (i). Q.E.D.

Example 4: *Risk aversion*

When the agent is risk averse, the renegotiation-proof value function can lie strictly below the Pareto frontier of the full-commitment value function. The following computed example illustrates this possibility. The stage model is the same as in Example 1 (Section 2), except that the agent is now risk averse toward income: his period utility function is given by $\sqrt{c} - a$, where c is consumption or income and a is the utility cost of taking action a. The RP and full-commitment value functions are computed and are shown in Figure 4. The RP value function lies below the Pareto frontier of the full-commitment value function.

[INSERT FIGURE 4 HERE]

6.3 Non-existence and non-convergence

Proposition 3 proves the general existence of principal-renegotiation-proof value function, which is a *nonincreasing* function. However, such a PRP value function may not be strictly decreasing and therefore may fail to be Self-Pareto-Generating or renegotiation-proof. In fact Self-Pareto-Generating value function may not even exist. The following example illustrates this possibility.

EXAMPLE 5

The stage model is identical to that in Example 2. The emphasis here will be on the principal-renegotiation-proof and full-commitment value functions. In the computation, ε is assumed to be 0.02. The lower bound of the PRP value function turns out to be $\ell^* = 1.74$. The value functions are shown in Figure 5.

[INSERT FIGURE 5 HERE]

The PRP value function is not strictly decreasing, as seen from the graph. In fact, one can show that a Self-Pareto-Generating value function can not exist for this example.

LEMMA 7. *There does not exist Self-Pareto-Generating value function for Example 5.*

Proof. Suppose to the contrary that there exists a Self-Pareto-Generating value function $f : [\ell^*, \bar{\xi}]$. It is straightforward to verify that given decreasing continuation value function f the peak point of the generated value function Γf occurs either at $\delta\ell^*$ (by implementing action a_1) or at $\delta\ell^* + r_2$ (by implementing a_2). In the former case, one must have $\ell^* = 0$, which is impossible. In the latter case, one must have $\ell^* = \delta\ell^* + r_2$ or $\ell^* = r_2/(1-\delta)$. But then the principal can get better payoff by implementing a_1 and offering agent a payoff equal to $\delta\ell^*$, contradicting that f is Self-Pareto-Generating. Q.E.D.

Consequently, if one repeatedly applies Pareto-generating operator Φ to some function f in this example, the sequence of functions $\langle \Phi^n f \rangle$ will not converge. In the next section, I provide sufficient conditions that guarantee convergence.

7 A Convergence Result

This section is concerned with the convergence of the sequence of functions $\langle \Phi^n(f) \rangle$ which are generated by continuously applying the Pareto generating operator Φ to a given value function f.

Let $\mathcal{F}$ denote the set of potential PRP functions. Formally, a real-valued continuous function f defined on a positive interval $[\ell, \bar{\xi}]$ belongs to $\mathcal{F}$ if (a) f is nonincreasing and concave; (b) for every $\xi_1 \neq \xi_2$ within $[\ell, \bar{\xi}]$:

$$\frac{f(\xi_1) - f(\xi_2)}{\xi_1 - \xi_2} \geq -M,$$

where $M > 0$ is given in Lemma 2; (c) for every $\xi \in [\ell, \bar{\xi}]$, $f(\xi) \leq V^f(\xi)$, where $V^f(\cdot)$ is the full-commitment value function (i.e. when the lower bound on the promised utility is zero.).

Let f_1 and f_2 be two functions in $\mathcal{F}$, with $f_1 : [\ell_1, \bar{\xi}] \longrightarrow \Re$ and $f_2 : [\ell_2, \bar{\xi}] \longrightarrow \Re$. For our purposes, the distance between f_1 and f_2 is defined as follows:

$$d(f_1, f_2) = \max\Bigg(|\ell_1 - \ell_2|, \sup_{\xi \in [\max(\ell_1, \ell_2), \bar{\xi}]} |f_1(\xi) - f_2(\xi)| \Bigg) \qquad (11)$$

Recall that the Pareto generating operator Φ maps $\mathcal{F}$ into $\mathcal{F}$ in the following manner: for any $f \in \mathcal{F}$, the function Φf is the (strictly) *decreasing* portion of Γf, the value function generated by f.

Let $f \in \mathcal{F}$. Successively applying operator Φ to the Pareto generated value functions, one obtains a sequence of functions, $\langle \Phi^n f \rangle$. There is one difficulty in showing that the sequence $\langle \Phi^n f \rangle$ converge: the domain of the generated functions can vary, which makes it hard to directly apply the usual fixed point theorems.

In Proposition 7 below, I derive a sufficient condition for $\langle \Phi^n f \rangle$ to converge. To prove the proposition, the following Lemma is needed, which describes a form of continuity of operator Φ: if two members of $\mathcal{F}$ get close, so do the two Pareto generated functions.

LEMMA 8. *Let $f_1, ..., f_n, ...$ be a sequence of functions in $\mathcal{F}$. Suppose there is a function f_0 in $\mathcal{F}$ such that $\lim_{n\to\infty} d(f_0, f_n) = 0$. Then*

$$\lim_{n\to\infty} d(\Gamma f_0, \Gamma f_n) = 0, \text{and} \lim_{n\to\infty} d(\Phi f_0, \Phi f_n) = 0.$$

Proof. See the appendix. Q.E.D.

It is now ready to state and prove the main result of this section, which shows that if the domains of the functions $\langle \Phi^n f_0 \rangle$ converge in Hausdorf metric, then $\langle \Phi^n f_0 \rangle$ converge.

PROPOSITION 7. *Let $f \in \mathcal{F}$. Let domain of $\Phi^n f$ be $[\ell_n, \bar{\xi}]$. Suppose $\langle \ell_n \rangle$ converge to $\ell^* \neq \bar{\xi}$. Then $\langle \Phi^n f \rangle$ converge to a strictly decreasing function $f^* \in \mathcal{F}$, i.e. $\lim_{n\to\infty} d(\Phi^n f, f^*) = 0$, and f^* is Self-Pareto-Generating: $\Phi f^* = f^*$. Moreover, if ℓ^* is independent of f, then the Self-Pareto-Generating value function f^* is unique.*

Proof. First, I show that there is a subsequence $\langle \Phi^{n_k} f \rangle$ that converge to some function $f^{**} : [\ell^*, \bar{\xi}] \longrightarrow \Re$. Extend each function $\Phi^n f$ to the entire interval $[0, \bar{\xi}]$ by letting $\Phi^n f(\xi)$ equal to the maximum of $\Phi^n f$ for each ξ that was not previously in its domain. Graphically, this amounts to horizontally extending the graph of $\Phi^n f$ to hit the vertical axis. It is straightforward to verify that the family of the extended functions $\langle \Phi^n f \rangle$ is equicontinuous. Then by Ascoli-Arzela Theorem (see [20], page 169.), there exists a subsequence $\langle \Phi^{n_k} f \rangle$ that uniformly converge to a continuous function f^{**}. Now the sequence $\langle \Phi^{n_k} f \rangle$ without extension converge to the restriction of f^{**} on $[\ell^*, \bar{\xi}]$, which will still be denoted by f^{**} to lessen notational burden.

Second, I show that there is a subsequence $\langle \Phi^{m_j} f \rangle$ that converge to the fixed point, $f^* : [\ell^*, \bar{\xi}] \longrightarrow \Re$, of the contraction mapping operator T defined in Eq. (5) (section 4.1), i.e. $Tf^* = f^*$. Fix an $\epsilon > 0$. Consider the sequence $\langle \Gamma(\Phi^{n_k} f) \rangle$, each term of which is obtained by applying Γ to a term in $\langle \Phi^{n_k} f \rangle$. By Lemma 8, $\langle \Gamma(\Phi^{n_k} f) \rangle$ converge to Γf^{**}. Now $\langle \Phi^{n_k+1} f \rangle$ converge to Tf^{**}, the restriction of

Γf^{**} to $[\ell^*, \bar{\xi}]$. Pick one element of the sequence and denote it by $\Phi^{m_1} f$, so that $d(\Phi^{m_1} f, Tf^{**}) \leq \frac{\epsilon}{2}$. Similarly, $\langle \Phi^{n_k+j} f \rangle$ converge to $T^j f^{**}$. I then pick $\Phi^{m_j} f$, so that $d(\Phi^{m_j} f, T^j f^{**}) \leq \frac{\epsilon}{2^j}$. Since $\langle T^j f^{**} \rangle$ converge to the fixed point $f^* = Tf^*$, the sequence $\langle \Phi^{m_j} f \rangle$ converge to f^*.

Next, I show that $\langle \Phi^n f \rangle$ converge to f^*. Since $\langle \Phi^{m_j} f \rangle$ converge to f^*, it follows $\langle \Phi^{m_j+1} f \rangle$ converge to $Tf^* = f^*$. By induction, $\langle \Phi^{m_j+k} f \rangle$ converges to f^*, for all $k = 1, 2, \cdots$. The union of these sequences, excluding repetitions of terms, differ from $\langle \Phi^n f \rangle$ for only finitely many elements. It follows that $\langle \Phi^n f \rangle$ converge to f^*.

Finally, by Lemma 8, $\langle \Phi(\Phi^n f) \rangle$ converge to $\Phi(f^*)$, which implies $f^* = \Phi f^*$. The last sentence of the proposition obviously holds. Q.E.D.

The condition that the domains of $\langle \Phi^n f \rangle$ converge is crucial for the proposition to hold. In general, the sequence of Pareto-generated functions $\langle \Phi^n f \rangle$ may not converge at all. See Example 5 in Section 6 for an illustration.

I close this section by presenting an intuitive definition of RP contracts and show that it coincides with the notion of Self-Pareto-Generating if the convergence result can be applied. The definition links the infinite-horizon setting to the finite-horizon setting.

A contract σ is 0-period renegotiation proof if σ is Pareto optimal. A contract σ is T-period renegotiation proof , for $T = 1, 2, ...$, if $\sigma | h^1$ is T-1-period renegotiation proof and σ is Pareto optimal among all incentive compatible and feasible contracts σ' whose continuation contracts $\sigma' | h^1$ are T-1-period renegotiation proof.

Let Q_T be the value set associated with T-period renegotiation-proof contracts. The intuitive definition for renegotiation-proofness would be ∞-period RP, which is associated with the limit of $\langle Q_T \rangle$. If the sequence of sets $\langle Q_T \rangle$ indeed converges, then the limit is a Self-Pareto-Generating (or weak renegotiation-proof) value function. So these two notions, ∞-period RP and Self-Pareto-Generating coincide.

8 The Case of Two Outputs and Two Actions

In this section, I analyze in detail a simple special case of the contracting problem: the two-action-two-outcome case. Throughout this section I assume that the agent has two actions, so $A = \{a_1, a_2\}$, with $g(a_2) > g(a_1) = 0$, and there are two output levels, i.e. $Y = \{y_1, y_2\}$ with $y_2 > y_1$. Let $p_h = \text{prob}(y_2 | a_2)$ and $p_\ell = \text{prob}(y_2 | a_1)$.

With this much simplified setting, more can be said about RP contracts. The main result in this section is to show that a unique fixed point exists for the Pareto Generating operator Φ, so a unique RP value function exists. As a result, there is a natural connection between RP value functions in finite-horizon setting and that in infinite-horizon setting, as seen at the end of last section, which allows a very intuitive interpretation of renegotiation-proofness.

The next a few lemmas show that in the two-action two-outcome case the condition required for Proposition 7 to apply is satisfied, therefore there exists a unique renegotiation-proof value function. The following assumption is needed, which is quite general yet sufficient to give rise to the desired result.

A 4. With the agent's reservation utility being zero, it is strictly optimal for the principal to induce action a_2 in the one-period contracting problem. (Note: This assumption implies $p_h > p_\ell$.)

LEMMA 9. *Suppose that assumption A4 holds. Given any nonincreasing continuation value function, it is strictly optimal for the principal to induce action* a_2.

Proof. See the appendix. Q.E.D.

Pick an arbitrary strictly decreasing concave function $f_0 : [\ell_0, \bar{\xi}] \longrightarrow \Re$ from set $\mathcal{F}$. Continuously applying operator Φ, one obtains a sequence Φf_0, $\Phi^2 f_0$, ..., $\Phi^n f_0, \ldots$. We are interested in whether the sequence converges to some function f^* with $\Phi f^* = f^*$. First, the next lemma shows that the domains of $\langle \Phi^n f_0 \rangle$ converge.

LEMMA 10. *Let f_0 be a strictly decreasing function in $\mathcal{F}$. For $n = 0, 1, \ldots$, let $[\ell_n, \bar{\xi}]$ be the domain of function $\Phi^n f_0$. Then there exists some $\ell^* \in [0, \bar{\xi}]$ such that $\langle \ell_n \rangle$ converges to ℓ^*.*

Proof: By lemma 9, for all n and $f_n \equiv \Phi^n f_0$, the maximum of function Γf_n is attained by inducing a_2. Let $(z_1, z_2), (\xi_1, \xi_2)$, where $z_1, z_2 \geq 0$ and $\xi_1, \xi_2 \in [\ell_n, \bar{\xi}]$, be a vector of current and promised utilities that attain the maximum of Γf_n. It follows that they must satisfy:

$$z_1 = 0; \xi_1 = \ell_n$$

$$(p_h - p_\ell)(z_2 + \delta\xi_2 - z_1 - \delta\xi_1) = g(a_2)$$

Thus the point ℓ_{n+1} is given by

$$\ell_{n+1} = p_h(z_2 + \delta\xi_2) + (1 - p_h)(z_1 + \delta\xi_1) - g(a_2)$$

which by the two preceding equations becomes

$$\ell_{n+1} = \delta\ell_n + \frac{p_\ell g(a_2)}{p_h - p_\ell}.$$

It follows that $\ell_n = \delta^n\ell_0 + (\delta^{n-1} + ... + \delta + 1)\frac{p_\ell g(a_2)}{p_h - p_\ell} = \delta^n\ell_0 + (1 - \delta^n)\ell^*$, where $\ell^* \equiv \frac{p_\ell g(a_2)}{(1-\delta)(p_h - p_\ell)}$. Hence $\ell_n \longrightarrow \ell^*$.

Q.E.D.

Now Lemma 10 and Proposition 7 imply the following main result for the two-action-two-outcome case.

PROPOSITION 8. *Suppose that assumption A4 holds. Then a unique RP value function exists.*

Remark: Lemma 8 and Proposition 7 (in section 5) do not depend on two-action two-outcome assumption. However, to apply these results to get unique RP value function, Pareto generating operator Φ need to produce a unique set of promised utilities, i.e. applied to any $f_0 \in \mathcal{F}$, $\lim_{n\to\infty} \Phi^n f_0$ should have the same domain. For two-action two-outcome case, this is easy to obtain.

9 Summary

This paper studies renegotiation-proof contracts in the infinite horizon principal-agent framework. The concept of renegotiation-proofness naturally extends the one used in finite-horizon settings. It proves fruitful for existence and characterization to allow public randomization over menu of contracts. The renegotiation-proof value function has a simple characterization: it is the optimal value function with an appropriate lower bound placed on the agent's promised utility. This result thus establishes the equivalence between renegotiation-proof contracts and optimal contracts with one-sided limited commitment on the part of the agent. I have identified sufficient conditions for renegotiation-proof value functions in finite-horizon settings to converge to a renegotiation-proof value function in infinite horizon, as time goes to infinity.

Appendix A: Proofs

Proof of Lemma 4. The proof consists of two parts.

Part I. Show left continuity at each $\ell \in (0, \bar{\xi}]$.

Step 1. The plan of the proof.

Let T be the contraction mapping operator embedded in functional equation (5) (in section 4.1) when ℓ is the lower bound on promised utility. Let f be the restriction of $V(\cdot, \ell')$ on $[\ell, \bar{\xi}]$. I will repeatedly apply operator T to function f and show that the distance between f and $T^n f$, for all n, is bounded by a constant multiple of $\gamma \equiv |\ell - \ell'|$. Since $T^n f$ converges to $V(\cdot, \ell)$, continuity will follow.

Step 2. $V(\cdot, \ell')$ has uniform bounded variation across all $\ell' (< \ell)$ that are close to ℓ.

By Lemma 2, the derivatives $V(\cdot, \ell')$ are bounded from below by some $-M$, so only need to find an upper bound.

Let $x \equiv \frac{1}{2}(\delta\ell + \ell)$. Let $V^f(\cdot) : [0, \bar{\xi}] \longrightarrow \Re$ be the full commitment value function. Let

$$K' \equiv \frac{V^f(x) - \Gamma V(\delta\ell, \ell)}{x - \delta\ell}.$$

Since by Lemma 3, $V^f(x) \geq \Gamma V(x, \ell')$ and $\Gamma V(\delta\ell, \ell') \geq \Gamma V(\delta\ell, \ell)$, one has

$$\frac{\Gamma V(x, \ell') - \Gamma V(\delta\ell, \ell')}{x - \delta\ell} \leq K'$$

Since $\Gamma V(\cdot, \ell')$ is concave, the derivative of $\Gamma V(\cdot, \ell')$ (hence that of $V(\cdot, \ell')$) at any $\xi \in [x, \bar{\xi}\,]$ is less than K'. Thus, for all $\ell' \in (x, \ell)$, the derivatives of $V(\cdot, \ell')$ at any $\xi \in [\ell', \bar{\xi}\,]$ is less than K'.

Let $K = \max\{M, |K'|\}$. Then for all $\ell' \in (x, \ell)$, the left and right derivatives of $V(\cdot, \ell')$ are contained in the interval $[-K, K]$.

Step 3. Find a lower bound on Tf.

For any $\xi \in [\ell, \bar{\xi}]$, let $(\pi^j, w^j, \xi^j, a^j)_{j \in J}$ be a solution that attains $V(\xi - \gamma, \ell')$, given continuation value function $V(\cdot, \ell')$:

$$V(\xi - \gamma, \ell') = \sum_j \pi^j \sum_y p(y|a^j) \left[y - u^{-1}(z^j(y)) + \delta V(\xi^j(y), \ell') \right], \qquad (12)$$

where each $z^j(y) = u^{-1}(w^j(y))$.

Now for each j, construct $\tilde{z}^j$ and $\tilde{\xi}^j$ such that for all $y \in Y$, $\tilde{z}^j(y) \geq z^j(y)$, $\tilde{\xi}^j(y) \geq \xi^j(y)$, $\tilde{\xi}(y) \geq \ell$ [13] and

$$\tilde{z}^j(y) + \delta\tilde{\xi}(y) = z^j(y) + \delta\xi^j(y) + \gamma. \tag{13}$$

It can be shown that for all $z = z^j(y)$ and $\tilde{z} = \tilde{z}^j(y)$,

$$\begin{aligned} -u^{-1}(\tilde{z}) &\geq -u^{-1}(z+\gamma) \qquad (14) \\ &\geq -\frac{du^{-1}(z+\gamma)}{dz}\gamma - u^{-1}(z) \\ &= -\frac{\gamma}{u'(u^{-1}(z+\gamma))} - u^{-1}(z) \\ &\geq -M\gamma - u^{-1}(z). \end{aligned}$$

where again $M > 0$ is given in Lemma 2.

Moreover, for each $y \in Y$, since $|\tilde{\xi}^j(y) - \xi^j(y)| \leq \gamma$ and $V(\cdot, \ell')$ has bounded variation by Step 2, it follows that

$$V(\tilde{\xi}^j(y), \ell') \geq V(\xi^j(y), \ell') - K\gamma. \tag{15}$$

Clearly, the vector $(\pi^j, \tilde{w}^j, \tilde{\xi}^j, a^j)_{j \in J}$ is feasible, incentive compatible, and promise the agent exactly the value ξ. By equations (12), (14) and (15), one concludes that for all $\xi \in [\ell, \bar{\xi}]$,

$$Tf(\xi) \geq V(\xi - \gamma, \ell') - K\gamma$$

which again by bounded variation implies that for $\xi \in [\ell, \bar{\xi}]$,

$$Tf(\xi) \geq V(\xi, \ell') - 2K\gamma.$$

Letting $B \equiv 2K$ and noticing f is the restriction of $V(\cdot, \ell')$ to $[\ell, \bar{\xi}]$, one has

$$f(\xi) \leq Tf(\xi) + B\gamma. \tag{16}$$

Step 4. Prove left continuity.

Since operator T is monotone and discounting, it follows

$$Tf(\xi) \leq T(Tf)(\xi) + \delta B\gamma,$$

[13] For some y, $\xi(y)$ may be in $[\ell', \ell)$.

which by (16) implies

$$f(\xi) \leq (T^2 f)(\xi) + B\gamma + \delta B\gamma.$$

Repeatedly applying operator T on both sides and regrouping terms, one obtains, for all n,

$$f(\xi) \leq T^n f(\xi) + B\gamma + \delta B\gamma + ... + \delta^{n-1} B\gamma.$$

Since $\lim_{n\to\infty} T^n f(\xi) = V(\xi, \ell)$, for all $\xi \in [\ell, \bar{\xi}]$, it follows

$$V(\xi, \ell') \leq \frac{B\gamma}{1-\delta} + V(\xi, \ell).$$

Since for all $\xi \in [\ell, \bar{\xi}]$, $V(\xi, \ell') \geq V(\xi, \ell)$, it follows

$$\left| V(\xi, \ell') - V(\xi, \ell) \right| \leq \frac{B\gamma}{1-\delta} = \frac{B}{1-\delta} |\ell - \ell'|.$$

This establishes left continuity at $\ell \in (0, \bar{\xi}]$.

Part II. Show right continuity at any $\ell \in [0, \bar{\xi})$.

The argument parallels that in Part I. Consider an $\ell' > \ell$. Let f be the restriction of $V(\cdot, \ell)$ on $[\ell', \bar{\xi}]$. Let T be the contraction mapping operator embedded in Eq. (5) when the lower bound on promised utility is ℓ'. The strategy of proof is again to keep applying operator T on f and show that the distance between f and $T^n f$ is bounded by some constant multiple of $|\ell - \ell'|$.

The only difference from Part I is the argument for bounded variation of function $V(\cdot, \ell)$, which in fact is much simpler to obtain in this case. A lower bounded $-M$ on the left and right derivatives of $V(\cdot, \ell)$ is already known to exist; concave function $V(\cdot, \ell)$ can be extended to a concave function $\Gamma V(\cdot, \ell)$ on $[\delta\ell, \bar{\xi}]$, which implies the existence of finite left derivative at ℓ; that is an upper bound. Thus there exists some $K > 0$ that bounds the absolute values of left and right derivatives of $V(\cdot, \ell)$ on $[\ell, \bar{\xi}]$.

The rest of the proof goes through similarly as in part I. Q.E.D.

Proof of Lemma 8. The following result will be used for the proof.

Claim: There exists a constant $K > 0$ such that the left and right derivatives of function Γf_0 are bounded within the interval $[-K, K]$.

To prove the claim, first note that by Lemma 2 there exists $M > 0$ such that the left and right derivatives of function Γf_0 are bounded from below by $-M$. To find an upper bound, let V_{max} be the maximum of function Γf_0. Let β be the minimum promised payoff to the agent while the principal is still able to implement an action a with $g(a) > 0$. Note that $\beta > \delta\ell_0$. Define $K' = (V_{max} - \Gamma f_0(\delta\ell_0))/(\beta - \delta\ell_0)$. Clearly, the left and right derivatives of function Γf_0 are bounded from above by K'. The claim follows by taking $K = \max\{K', M\}$.

To proceed, for each $n = 0, 1, ...$, let the domain of f_n be $[\ell_n, \bar{\xi}]$; let $\gamma_n \equiv |\ell_n - \ell_0|$; let $d_n \equiv d(f_0, f_n) \geq \gamma_n$. Note that $\gamma_n \to 0$ and $d_n \to 0$.

I will show that there exists some $B > 0$ such that for all n, $|\Gamma f_n(\xi) - \Gamma f_0(\xi)|$ is uniformly bounded on the interval $[\max(\delta\ell_n, \delta\ell_0), \bar{\xi}\,]$ by Bd_n. The conclusion of the lemma will then follow from there.

First consider the case when $\ell_n < \ell_0$. Fix $\xi \in [\delta\ell_0, \bar{\xi}\,]$. Let $(\pi^j, w^j, \xi^j, a^j)_{j\in J}$ be a vector that attains value $\Gamma f_0(\xi)$. It follows that

$$\Gamma f_0(\xi) = \sum_{j\in J} \pi^j \sum_y p(y|a^j) \left\{y - w^j(y) + \delta f_0(\xi^j(y))\right\}$$

$$\leq \sum_{j\in J} \pi^j \sum_y p(y|a^j) \left\{y - w^j(y) + \delta[f_n(\xi^j(y)) + d_n]\right\}$$

$$\leq \Gamma f_n(\xi) + d_n.$$

On the other hand, let $(\pi_i, w^i(\cdot), \xi^i(\cdot), a^i)_{i\in I}$ be a vector that generates $\Gamma f_n(\xi)$. Then

$$\Gamma f_n(\xi) = \sum_{i\in I} \pi^i \sum_y p(y|a^i) \left\{y - w^i(y) + \delta f_n(\xi^i(y))\right\}$$

$$\leq \sum_{i\in I} \pi^i \sum_y p(y|a^i) \left\{y - w^i(y) + \delta \left[f_n(\xi^i(y) + \gamma_n) + M\gamma_n\right]\right\}$$

$$\leq \sum_{i\in I} \pi^i \sum_y p(y|a^i) \left\{y - w^i(y) + \delta \left[f_0(\xi^i(y) + \gamma_n) + d_n\right]\right\} + M\gamma_n$$

$$\leq \Gamma f_0(\xi + \gamma_n) + (M + 1)d_n$$

$$\leq \Gamma f_0(\xi) + K\gamma_n + (M + 1)d_n$$

$$\leq \Gamma f_0(\xi) + (K + M + 1)d_n.$$

The first inequality follows because the left and right derivatives of function f_n are within $[-M, 0]$; the second follows from the fact that $d(f_n, f_0) \to 0$; the third follows from the suboptimality of menu $(\pi^i, w^i, \xi^i + \gamma_n, a^i)_{i\in I}$; the fourth follows from the claim in the above.

Similar result holds if $\ell_n > \ell_0$. In conclusion, this shows that there exists some B such that $|\Gamma f_n(\xi) - \Gamma f_0(\xi)| < Bd_n$ for all $\xi \in [\max(\delta\ell_n, \delta\ell_0), \bar{\xi}\,]$. Hence $d(\Gamma f_n, \Gamma f_0) \to 0$.

As for operator Φ, let a subsequence $\langle \Phi f_{n_k} \rangle$ converge to a function $h : [\ell', \bar{\xi}] \longrightarrow \Re$. Let $\Phi f_0 : [\hat{\ell}, \bar{\xi}] \longrightarrow \Re$. Note that Φf_0 and h coincide on $[\max(\ell', \hat{\ell}), \bar{\xi}]$. Suppose $\hat{\ell} \neq \ell'$. If $\hat{\ell} < \ell'$ then

$$\Gamma f_0(\hat{\ell}) = \Phi f_0(\hat{\ell}) > \Phi f_0(\ell') = h(\ell') \geq \lim_{n_k \to \infty} \Gamma f_{n_k}(\hat{\ell}) = \Gamma f_0(\hat{\ell}),$$

which is a contradiction. If $\hat{\ell} > \ell'$, a similar contradiction is reached. Thus $d(\Phi f_0, h) = 0$. Q.E.D.

Proof of Lemma 9. First, consider the one-period problem. The maximum return the principal can get by inducing a_1 is:

$$E_1 y \equiv p_\ell y_2 + (1 - p_\ell) y_1.$$

The maximum return the principal can get by inducing a_2 is:

$$E_2 y - p_h w_2^* \equiv p_h y_2 + (1 - p_h) y_1 - p_h w_2^*,$$

where wage scheme $(w_1^* = 0, w_2^*)$ is such that,

$$(p_h - p_\ell) u(w_2^*) - g(a_2) = 0.$$

So assumption A4 amounts to

$$E_2 y - p_h w_2^* > E_1 y. \tag{17}$$

Fix a nonincreasing function $f : [\ell, \bar{\xi}] \longrightarrow \Re$. Consider the generating problem, as defined in section 3. One way to induce action a_2 is to use current wage scheme $(w_1^* = 0, w_2^*)$ and promise continuation utility ℓ regardless of the outcome. The return is:

$$E_2 y + \delta f(\ell) - p_h w_2^*,$$

which by Eq. (17) is strictly bigger than the maximum return by inducing a_1: $E_1 y + \delta f(\ell)$. Q.E.D.

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