Renegotiation-Proof Contract in Repeated Agency

Rui R. Zhao[*]
Department of Economics
University at Albany - SUNY
Albany, NY 12222, USA
E-mail: rzhao@albany.edu
Tel: 518-442-4760
Fax: 518-442-4736

This version: April 2005

Running title: Renegotiation-proof Contract

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[*] I thank Narayana Kocherlakota, Michael Jerison, and seminar participants at SUNY-Albany, University of Minnesota, 2002 Meetings of the Society of Economic Dynamics, and 2003 North-America Summer Meetings of the Econometric Society for helpful comments. I am especially grateful to an anonymous referee and the associate editor for their detailed and insightful comments, which helped improve the paper significantly. All remaining errors are my own.

Abstract

Renegotiation-proof contracts are studied in infinitely repeated principal-agent problem. Contracts satisfying a weaker notion of renegotiation-proofness always exist. The renegotiation-proof value function has a simple characterization: It is the principal's optimal value function when an appropriate lower bound is placed on the agent's expected utility. Sufficient conditions are provided for renegotiation-proof value function in finite horizon to converge to renegotiation-proof value function in infinite horizon as time goes to infinity. *Journal of Economic Literature* Classification Numbers: D8, C7

Keywords: Renegotiation Proof Contracts, Repeated Principal-Agent

1 Introduction

In repeated principal-agent problem the principal may be able to offer the agent a long-term contract but unable to commit not to renegotiate it in the future. This is important because for incentive reasons ex ante efficient full-commitment contracts may be ex post inefficient hence will be subject to renegotiation. The effective contract therefore has to be *renegotiation proof*. Wang [27] characterizes renegotiation-proof contracts in a model of finitely repeated moral hazard. This paper goes one step further to study renegotiation-proof contracts in infinite-horizon principal-agent problem.

Renegotiation-proofness is well understood at the conceptual level for finite horizon (see Benoit and Krishna [2] and Wang [27]). The idea can be summarized as follows. Starting from the last period, one-period renegotiation-proof contracts are simply Pareto optimal contracts. Using backward induction a T-period renegotiation-proof contract is defined to be Pareto optimal subject to the condition that the continuations of the contract are T-1-period renegotiation-proof. This backward induction method, however, does not work for infinite horizon. As a result multiple renegotiation-proof concepts have been proposed for infinite horizon.[1] In this paper I adopt the concept proposed by Ray [19], which naturally generalizes the backward induction definition for finite horizon. At this point, roughly speaking a contract is renegotiation-proof if it is Pareto optimal within the class of contracts whose continuation contracts are renegotiation-proof.

To illustrate the idea it is helpful to consider the utility frontier generated by full-commitment contracts, which represents the maximum utility of the principal for each given utility level of the agent. A contract is efficient if the generated utility pair is not Pareto dominated by any other pair on the frontier. As pointed out by Fudenberg et al [7], if the continuation utility frontier at every point in time is downward-sloping (which would be the case if the agent's utility function is unbounded below) then ex ante efficient full-commitment contracts are always ex post efficient therefore are renegotiation-proof.[2]

In infinitely repeated principal-agent, the utility frontier is invariant over time. The analysis of this paper will focus on the situation in which the frontier is not downward-sloping so renegotiation potentially matters. Specifically in the model, as in Wang [27], the agent has limited liability which requires a minimum wage. By the argument of the efficiency-wage theory (e.g. Shapiro and Stiglitz [23]), to induce nontrivial effort the principal needs to pay the agent over and above his reservation utility, i.e. the agent will receive positive rent. In dynamic environment the principal can optimally structure the intertemporal wage payments so as to minimize the agent's rent. Such ex ante optimal wage contracts however may require the principal to commit to some ex post inefficient continuation contracts. In other words the utility frontier is not entirely downward-sloping, which makes renegotiation relevant whenever

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[1] See among others, Bernheim and Ray [4], Farrell and Maskin [6], van Damme [26], Abreu, Pearce, and Stacchetti [1], Bergin and MacLeod [3], Ray [19], and Kocherlakota [12]. Bergin and MacLeod also discuss the relationships between various concepts.

[2] In a related paper Rey and Salanie [20] show that short-term contracts with renegotiation can achieve full-commitment long-term efficiency if the range of one-period wage payments is sufficiently large.

the continuation utility pair falls on the upward-sloping portion of the frontier.

Renegotiation-proofness requires that the utility frontier be *Self-Pareto-Generating*, i.e. the Pareto frontier of current utility frontier is the same as the continuation utility frontier. The main result of this paper is a simple characterization of the most efficient Self-Pareto-Generating utility frontier, which is shown to be the principal's optimal value function when an appropriate lower bound is placed on the agent's expected utility. This result generalizes Wang's [27] finding for finitely repeated moral hazard. Wang shows that efficient full-commitment contracts may not be renegotiation-proof if the agent's expected utility is too low. The result here is sharper: It is shown that there is a cutoff in the agent's expected utility above which efficient contracts are renegotiation-proof below which they are not renegotiation-proof. Moreover, this cutoff is explicitly constructed using a simple algorithm. I also show that finite-horizon renegotiation-proof value functions converge to an infinite-horizon renegotiation-proof value function as time goes to infinity. Using this convergence result I show that in the case of two actions and two output levels there exists a unique renegotiation-proof value function.

The above characterization results imply that renegotiation-proof value function is closely related to the utility frontier generated by efficient *limited-commitment* contracts in which the agent can walk away and receive some default payoff from an outside option (see Phelan [16] and Thomas and Worrall [25] for contracting with limited commitment, and Ljungqvist and Sargent [14] for a survey of applications). However, one should note that the two settings are different: With limited commitment the agent can unilaterally walk away from the ongoing contract; with renegotiation there must be mutual consent for any change in the contract.

The analysis in this paper can be contrasted with the literature on contracting with renegotiation under asymmetric information. Fudenberg and Tirole [8] and Park [15] analyze the principal-agent problem in which the agent has private information at the renegotiation stage so the principal has to deal with adverse selection. In dynamic models of hidden information Dewatripont [5] and Laffont and Tirole [13] analyze long-term renegotiation-proof contracts, and Hart and Tirole [10] and Rey and Salanie [21] analyze links between long-term renegotiation-proof contracts and short-term limited-commitment contracts. Unlike these studies, in this paper at each renegotiation stage the principal and the agent always have symmetric information about their preferences over subsequent contingent outcomes, therefore renegotiation is not due to asymmetric information revelation but rather reflects the strategic value of principal's commitment to ex post inefficient contracts. Finally, along the lines of incomplete contracting Hermalin and Katz [11] study principal-agent in which there is an unverifiable common signal about the agent's effort choice at the renegotiation stage.

The rest of the paper is organized as follows. Section 2 provides a simple example to motivate the idea. Section 3 spells out the details of the model and introduces the concept of renegotiation-proofness. Section 4 presents the main characterization result and offers several remarks. Section 5 studies the link between finite and infinite horizons and provides a sharper result for the two-action two-outcome case. All proofs are contained in the Appendix.

2 An Example

To motivate idea consider the following two-period principal-agent example. In each period the agent can choose from two possible hidden actions a_2, a_1, with $a_2 > a_1 = 0$. There are two possible output levels $y_2 > y_1 = 0$. For $i = 1, 2$, let p_i be the probability of output y_2 occurring given action a_i. The agent has utility function $w - a$ where w is wage payment; the principal has utility function $y - w$. Both parties maximize expected discounted utility. Wage payments are nonnegative due to limited liability. The agent's reservation utility is normalized to zero.

Consider first the one-period problem. The principal can implement the agent's action choice by offering output-contingent wages. It is readily seen that the optimal contract for the principal to implement a_2 is to pay wage $w^* \equiv a_2/(p_2 - p_1)$ if output is y_2 and zero otherwise. Let R_i be the maximum return to the principal when action a_i is implemented; let r_i be the associated return to the agent. The parameters can be chosen so that it is jointly optimal to implement action a_2, i.e. $R_2 > R_1 \geq 0$, $r_2 > r_1 = 0$, and the following condition holds:[3]

$$p_2 r_2 - (1 - p_2)(R_2 - R_1) > 0. \tag{1}$$

Next consider the two-period model. The optimal contract for the principal now is history-dependent and given as follows: If output in period 1 is y_2 then pay current wage $w^* - \delta r_2$ and promise the agent the optimal one-period contract for period 2; otherwise, pay zero wage today and promise a flat zero wage for period 2. The upshot is that action a_2 is implemented in period 1; in period 2, a_2 is implemented if output in period 1 is y_2, otherwise a_1 is implemented. The principal's expected utility from this contract is given by

$$(1 + \delta)R_2 + \delta[p_2 r_2 - (1 - p_2)(R_2 - R_1)]$$

which by condition (1) is larger than $(1+\delta)R_2$, the utility received by repeating the one-period optimal contract. The intuition is as follows. Given nonnegativity of wages, the agent receives a positive rent r_2 when action a_2 is implemented. The principal can commit to implement inefficient action a_1 in period 2 if output in period 1 is low. By doing so the principal risks losing additional return $\delta(R_2 - R_1)$ with probability $1 - p_2$ but can save wage payment today by δr_2 with probability p_2. Condition (1) guarantees that this plan is profitable ex ante. But clearly if output in period 1 is indeed low then the principal and the agent will want to renegotiate the inefficient second-period contract. The same idea extends to infinite horizon.

3 The Model

The model is an infinite repetition of the standard principal-agent model. Time is discrete: t = 1, 2, ... At the beginning of time, the principal and the agent sign a long-term contract that specifies nonnegative wage payments to the agent contingent upon his past and current performances. The contract may be renegotiated at the start of every period. In each period the agent chooses an hidden action a_t from a finite set A. The set of possible output levels in

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[3] For example choose $p_2 = 0.8, p_1 = 0.2, y_2 = 1, a_2 = 0.3$.

every period is given by $Y = \{0 \leq y_1 < \cdots < y_I\}$. Each action $a \in A$ induces a probability distribution on set Y. Let $p_i(a)$ be the probability that output y_i will occur given action a.

I assume that the agent maximizes expected discounted utility given by

$$E\sum_{t=1}^{\infty} \delta^{t-1}(u(w_t) - g(a_t))$$

where $w_t \in \Re_+$ is wage payment and $a_t \in A$ is action choice in period t. The principal also maximizes expected discounted utility which is given by

$$E\sum_{t=1}^{\infty} \delta^{t-1}(y_t - w_t).$$

I make the following assumptions throughout the paper.

A 1. The agent's period utility function satisfies $u'(\cdot) > 0$, $u''(\cdot) \leq 0$, $u(0) = 0$, $g(a) \geq 0$ for all $a \in A$, and there exists an $\underline{a} \in A$ such that $g(\underline{a}) = 0$.

A 2. (Full Support) For all $a \in A$, $p_i(a) > 0$ for all $i = 1, ..., I$.

Public randomization of contracts will be permitted as this will simplify the analysis. In general let θ_t be a public signal drawn at the beginning of period t from the unit interval [0, 1] according to the uniform distribution. The public history at the end of period t then is given by $h^t = (y_1, ..., y_t; \theta_1, ..., \theta_t)$, i.e. the past realizations of outputs and public signals. A long-term contract consists of a wage plan which specifies wage payment $w(h^t)$ for all $t = 1, 2, ...$ and all history h^t, and an action plan which specifies action $a(h^{t-1}, \theta_t)$, $\forall\, t$, $\forall\, h^{t-1}$ and $\forall\, \theta_t$.

Since the agent's actions are hidden, contracts need be incentive compatible. For a given contract, in each period t let $\xi(h^{t-1})$ be the *continuation payoff* that the agent receives if the remainder of the contract is carried out from history h^{t-1} onwards. Then given history h^{t-1} and current signal $\theta_t = \theta$, the ongoing contract can always be decomposed into two components, a vector of current contingent wage payments ($w_i^\theta \equiv w(h^{t-1}, \theta, y_i)$) and a vector of continuation payoffs ($\xi_i^\theta \equiv \xi(h^{t-1}, \theta, y_i)$) "promised" to the agent. With this formulation, the contract is said to be *incentive compatible* if $\forall t$, $\forall h^{t-1}$, $\forall \theta$, given vectors (w_i^θ) and (ξ_i^θ), the agent's current action $a^\theta \equiv a(h^{t-1}, \theta)$ maximizes his expected payoff, i.e.

$$\sum_i [p_i(a^\theta) - p_i(a)]\big(u(w_i^\theta) + \delta \xi_i^\theta\big) \geq g(a^\theta) - g(a),\ \forall a \in A.$$

Next consider the feasibility of contract. A contract is said to be *feasible* if the continuation payoff of the agent at every history lies in interval $[0, \bar{\xi}\,]$ where $\bar{\xi} > 0$ is some given constant. The lower bound zero reflects limited liability of the agent: Since wage can not be negative and the agent can choose an action with zero utility cost, then the agent's expected utility can not go below zero. The upper bound $\bar{\xi}$ reflects certain kind of limited liability of the principal: Although there is no explicit limit on wages, there is a limit on how great the agent's expected utility can be. This formulation is partly for convenience, and it has the following motivation: In practice an employer rarely sets explicit upper limit on wages; she has the ability to borrow and pay wages beyond current revenue. Limited liability for the employer comes more from the fact that she may close her business if expected discounted profits fall below a threshold (say zero). Imposing an upper bound on the agent's payoff is a shortcut to this effect.

It is now ready to discuss the optimal contracting problem. The focus will be on contracts that are "renegotiation-proof." In particular I adopt the renegotiation-proof concept put forward by Ray [19]. For this purpose, it is useful to reformulate the contracting problem as the following *generating* problem. In a given period, suppose the principal's maximum continuation payoff through optimal contracting from next period onwards is represented by a value function f, i.e. $f(\xi)$ is the optimal continuation payoff of the principal given continuation utility ξ for the agent. Then in the current period the principal just needs to choose a vector of wage payments and a vector of continuation payoffs for the agent so that the agent will be induced to take some action that maximizes the principal's expected payoff. The solution is a new value function *generated* by f.

Formally, let $f : [\ell, \bar{\xi}] \to \Re$, where $0 \le \ell \le \bar{\xi}$, be the principal's continuation value function. The *generated* value function $\Gamma(f) : [\delta\ell, \bar{\xi}] \longrightarrow \Re$ is defined by

$$\forall \xi \in [\delta\ell, \bar{\xi}], \;\; \Gamma f(\xi) = \max_{(\pi^j, a^j, w_i^j, \xi_i^j)_{j \in J}} \sum_{j \in J} \pi^j \sum_i p_i(a^j) \left[y_i - w_i^j + \delta f(\xi_i^j) \right] \tag{2}$$

s.t.

$$\sum_{j \in J} \pi^j \left\{ \sum_i p_i(a^j) \left[u(w_i^j) + \delta \xi_i^j \right] - g(a^j) \right\} = \xi, \tag{3}$$

$$\sum_i \left[p_i(a^j) - p_i(a) \right] \left[u(w_i^j) + \delta \xi_i^j \right] \ge g(a^j) - g(a), \; \forall j \in J, \; \forall a \in A, \tag{4}$$

where $\forall j \in J, \forall i = 1, ..., I, \; w_i^j \in \Re_+, \; \xi_i^j \in [\ell, \bar{\xi}], \; a^j \in A, \; \pi^j \ge 0,$ and $\sum_{j \in J} \pi^j = 1$.

Several explanations are in order. Firstly, the two constraints in the problem are the "promise-keeping" constraint and the incentive constraint, respectively. The former guarantees payoff ξ to the agent. Secondly, the principal can randomly choose a contract from a menu $((w_i^j), (\xi_i^j), a^j)_{j \in J}$ with each contract j being chosen with probability π^j. As a computational procedure, the principal can first compute the optimal value function for implementing each implementable action without randomization and then take the upper concave frontier of all the value functions, which is the value function Γf. It is clear that in the current model this convexification only requires a menu of at most two different contracts. Thirdly, the generated function $\Gamma(f)$ is concave. Finally, the domain of $\Gamma(f)$ is $[\delta\ell, \bar{\xi}]$. Since wages are nonnegative and continuation payoffs are no less than ℓ, then the agent's minimum payoff is equal to $\delta\ell$. Randomization guarantees that any payoff within $[\delta\ell \; \bar{\xi}]$ is achievable.

Next define Φ as the *Pareto generating operator* so that $\Phi(f)$ is the Pareto frontier of the generated value function Γf. With this formulation, the main concept of renegotiation-proofness can now be given as follows.

Definition 1. A function $f : [\ell, \bar{\xi}] \to \Re$, where $0 \le \ell \le \bar{\xi}$, is *Self-Pareto-Generating* if the Pareto frontier of the generated value function $\Gamma(f)$ is identical to f, i.e. if $\Phi(f) = f$.[4] A contract is *renegotiation-proof* (RP) if the set of continuation utility pairs at all histories is a subset of the graph of a Self-Pareto-Generating value function.[5]

Self-Pareto-Generating captures the intuitive idea that a contract is renegotiation-proof if it is Pareto optimal subject to the condition that its continuation contracts are renegotiation-proof. This concept however can be too demanding when it comes to existence. A weaker concept, given below, will guarantee existence.

Definition 2. A nonincreasing function $f : [\ell, \bar{\xi}\,] \to R$ with $\ell \in [0, \bar{\xi}\,]$ is *semi-renegotiation-proof* (Semi-RP) if $\Gamma f(\xi) = f(\xi)$, $\forall \xi \in [\ell, \bar{\xi}\,]$ and $f(\ell) \geq \Gamma f(\xi)$, $\forall\, \xi \in [\delta\ell, \bar{\xi}\,]$.[6]

In other words, the graph of f can not contain any strictly increasing portion but it may contain a flat portion. Therefore it may be possible to make the agent strictly better off without hurting the principal but not the reverse. One justification for this concept might be that the principal is in the position to initiate renegotiation and can commit not to open renegotiation unless there is strict gain for herself; such commitment can be valuable for the principal and is credible since the agent can not bribe the principal through transfer due to limited liability (the agent's limited liability constraint binds whenever such commitment matters). Apparently, RP implies Semi-RP, but the reverse is not true.

4 Existence and Characterization

To begin, consider a class of optimal contracting problems in which the agent's continuation utilities are restricted to be within $[\ell, \bar{\xi}\,]$ for some $\ell \in [0, \bar{\xi}\,]$. For each promised utility $\xi \in [\ell, \bar{\xi}\,]$, let $V(\xi, \ell)$ be the maximum payoff of the principal. Then following standard argument,[7] the optimal value function $V(\cdot, \ell) : [\ell, \bar{\xi}\,] \longrightarrow \Re$ satisfies the following functional equation:

$$V(\xi, \ell) = \max_{(\pi^j, w_i^j, \xi_i^j, a^j)} \sum_{j \in J} \pi^j \sum_i p_i(a^j) \left[y_i - w_i^j + \delta V(\xi_i^j, \ell) \right] \tag{5}$$

subject to promise-keeping constraint (3) and incentive constraint (4).

Public randomization of contracts again is permitted, so $V(\cdot, \ell)$ is concave. Note that when $\ell = 0$ one obtains the full-commitment value function $V^f(\cdot) \equiv V(\cdot, 0)$. Note also the difference between this problem and the generating problem in the previous section: the domain of generated function is larger. The main result of this paper is given as follows.

Proposition 1. *Let set* $L = \{\, \ell \in [0, \bar{\xi}\,] : V(\cdot, \ell)$ is nonincreasing$\,\}$ *and* $\ell^* = \inf L$. *Then value function* $V(\cdot, \ell^*)$ *is semi-renegotiation-proof; if* $V(\cdot, \ell^*)$ *is strictly decreasing, then it is renegotiation-proof; moreover,* $V(\cdot, \ell^*)$ *weakly Pareto dominates any other Semi-RP function.*[8]

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[4] Ray [19] calls such function "internally renegotiation proof." Bergin and MacLeod [3] also introduce a similar concept, called "full recursive efficiency."

[5] In the rest of the paper, renegotiation-proof and Self-Pareto-Generating are used interchangeably.

[6] When $\ell = \bar{\xi}$, function f is defined on the singleton set $\{\bar{\xi}\}$ and is trivially nonincreasing.

[7] For instance see Green [9] and Spear and Srivastava [24].

[8] Namely, every utility pair on a given Semi-RP function is weakly dominated by a pair on $V(\cdot, \ell^*)$.

Note that L is nonempty since $\bar{\xi} \in L$. Note also that if $f : [\ell, \bar{\xi}] \to \Re$ is a RP or Semi-RP function then $\ell \in L$ and $f = V(\cdot, \ell)$, but the reverse is not true. Therefore by Lemma A.3 $V(\cdot, \ell^*)$ weakly Pareto dominates any other Semi-RP or RP function. Without further restriction on renegotiation one expects the two parties to reach this most efficient Semi-RP value function. Thus $V(\cdot, \ell^*)$ may be called *the* (semi)-renegotiation-proof value function.

Proposition 1 generalizes Wang's [27] finding for finite-horizon principal-agent. Wang shows that if the agent is promised an expected utility that is sufficiently low then the contract can not be renegotiation-proof. The result here sharpens Wang's finding by explicitly constructing a cutoff in the expected utility of the agent and showing that the contract is not RP if continuation utilities are below the cutoff and is RP if they are above the cutoff. The intuition for these results, as seen in Example 1, is as follow. At any point in time, if the agent's continuation utility is too low then the nonnegativity constraints on wages will be binding for some states. This can be counterproductive at ex post because by increasing utility of the agent the principal can relax these constraints and provide incentives more efficiently, thereby obtaining higher continuation payoff. However this kind of suboptimal schemes can be effective at ex ante because it allows the principal to cut back the rent the agent receives.

Remark 1. It is of interest to know whether the only Semi-RP function is degenerate, i.e. $\ell^* = \bar{\xi}$. It turns out that for practically all interesting problems, Semi-RP functions are nondegenerate: As long as in the one-period problem the Pareto frontier of the optimal value function is nondegenerate then $\ell^* < \bar{\xi}$, because in this case the degenerate function $V(\bar{\xi}, \bar{\xi})$ is not Semi-RP. On the other hand, if in the one-period problem it is strictly optimal for the principal to implement some action $\hat{a}$ with $g(\hat{a}) > 0$ then the full-commitment value function $V^f = V(\cdot, 0)$ is not nonincreasing, therefore $\ell^* > 0$.

Remark 2. Consider the relationship between Semi-RP value function $V(\cdot, \ell^*)$ and the Pareto frontier of full-commitment value function $V^f(\cdot)$. If both the principal and the agent are risk neutral, then it can be shown that there is a critical level of promised payoff for the agent above which the two functions coincide, so efficient full-commitment contracts are renegotiation-proof; moreover the payoff outcomes of such efficient contracts can be achieved using stationary one-period contracts. This result generalizes a finding by Fudenberg et al [7], who showed that the same type of result holds if the agent has full bargaining power (then the agent's expected payoff is above the critical level). It should be pointed out that in general the Pareto frontier of value function V^f is "larger" than $V(\cdot, \ell^*)$, namely there are efficient full-commitment contracts that are not renegotiation-proof even if both agents are risk neutral, as illustrated by Example 1 in Section 2.[9] On the other hand, if the agent is risk averse then value function $V(\cdot, \ell^*)$ can lie strictly below the Pareto frontier of the full-commitment value function, as illustrated by the following example:

Example 2. Same as Example 1 except that the agent's utility function is given by $\sqrt{w} - a$. The value functions are plotted in Figure 1.

Remark 3. Proposition 1 also establishes a connection between RP contracts and optimal

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[9] The infinitely repeated version of the example can be computed to illustrate this point.

contracts when the agent has limited commitment. With limited commitment, at the start of each period the agent can walk away and take some outside option thereafter. If $\ell \geq 0$ represents the agent's payoff from the outside option then the optimal value function is given by $V(\cdot, \ell)$. In general function $V(\cdot, \ell)$ is not nonincreasing for $\ell = 0$, but it may become nonincreasing for large enough ℓ. In the dynamic contracting literature it is informally argued that the downward-sloping value function generated by the smallest such lower bound corresponds to some "renegotiation-proof" contracts.[10] Proposition 1 validates this treatment by explicitly showing that value function $V(\cdot, \ell^*)$ indeed is renegotiation-proof. However one should note that despite this connection the primitives of the two environments are different: With limited commitment the agent can unilaterally walk away and the agent's payoff is exogenously constrained to be above ℓ^*; with renegotiation however there must be mutual consent for change in contract and the lower bound ℓ^* on agent's payoff is endogenous.

Remark 4. Allowing randomization and weakening RP to Semi-RP are essential since RP value function may not exist. This is illustrated by the following example.

EXAMPLE 3: There are two actions and two outputs: $a_1 = 0$, $a_2 = 0.2 + \varepsilon$, where $\varepsilon \geq 0$; $y_1 = 0$, $y_2 = 1$; $p(y_2|a_1) = 0.4$, $p(y_2|a_2) = 0.8$; agents are risk neutral. Recall that R_i and r_i are the one-period payoffs for the principal and the agent respectively when a_i is implemented optimally for the principal. It is straightforward to show that $r_1 = 0, R_1 = 0.4$, $r_2 = 0.2 + \varepsilon$, and $R_2 = 0.4 - 2\varepsilon$. Therefore the one-period optimal value function is decreasing for $\varepsilon > 0$.[11]

The infinite-horizon value functions are displayed in Figure 2, for parameter values $\bar{\xi} = 0.4$ and $\varepsilon = 0.02$. As seen from the graph, the Semi-RP value function has a flat portion. In fact, one can prove that a RP value function can not exist as follows. Suppose there exists a RP value function $f : [\ell^*, \bar{\xi}\,] \to \Re$ for some $\ell^* \geq 0$. It is straightforward to verify that given decreasing function f the peak of the generated function $\Gamma(f)$ occurs either at $\delta\ell^*$ (by implementing action a_1) or at $\delta\ell^* + r_2$ (by implementing a_2). In the former case, one must have $\ell^* = 0$, which is impossible since the full-commitment value function is not nonincreasing. In the latter case, one must have $\ell^* = \delta\ell^* + r_2$ or $\ell^* = r_2/(1-\delta)$, which implies payoff $R_2/(1-\delta)$ for the principal (by implementing a_2 throughout). But the principal can do better to implement a_1 for one period by offering zero wage and continuation payoff ℓ^*, which implies that the peak of $\Gamma(f)$ occurs at a point no larger than $\delta\ell^*$, a contradiction.

5 A Convergence Result and the 2×2 Case

This section is concerned with the convergence of the sequence of functions $\langle \Phi^n(f) \rangle$ which are obtained by repeatedly applying the Pareto generating operator Φ to a given value function f. For this purpose, define the distance between two continuous functions $f_1 : [\ell_1, \bar{\xi}\,] \to \Re$ and

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[10] For instance, Phelan and Townsend [17] hint at such a treatment; Quadrini [18] also uses a similar treatment to study renegotiation-proof contracts between a firm and an investor.

[11] This example shows that full-commitment value function may not be nonincreasing even if one-period value function is decreasing.

$f_2 : [\ell_2, \bar{\xi}\,] \rightarrow \Re$ as follows:

$$d(f_1, f_2) = \max\bigg(|\ell_1 - \ell_2|, \sup_{\xi \in [\max(\ell_1,\ell_2),\, \bar{\xi}\,]} |f_1(\xi) - f_2(\xi)|\bigg) \tag{6}$$

There is one difficulty in proving the convergence of sequence $\langle \Phi^n f \rangle$: The domains of these functions can vary which makes it hard to apply the usual fixed point theorems. Nevertheless the following proposition derives a sufficient condition for $\langle \Phi^n f \rangle$ to converge.

PROPOSITION 2. *Let* $\ell_0 \geq 0$ *and* $f : [\ell_0, \bar{\xi}\,] \rightarrow \Re$ *be continuous and* $f(\xi) \leq V^f(\xi)$, $\forall \xi \in [\ell_0, \bar{\xi}\,]$. *Let domain of* $\Phi^n f$ *be* $[\ell_n, \bar{\xi}\,]$, *for each* n. *Suppose* $\langle \ell_n \rangle$ *converges to some* $\ell^* \neq \bar{\xi}$. *Then* $\langle \Phi^n f \rangle$ *converges to a strictly decreasing function* $f^* : [\ell^*, \bar{\xi}\,] \rightarrow \Re$, *i.e.* $\lim_{n\rightarrow\infty} d(\Phi^n f, f^*) = 0$, *and* f^* *is Self-Pareto-Generating:* $\Phi f^* = f^*$. *Moreover, if* ℓ^* *is independent of* f, *then the Self-Pareto-Generating value function* f^* *is unique.*

The condition that the sequence of domains of functions $\langle \Phi^n f \rangle$ converges is crucial for the Proposition to hold. In general, the sequence $\langle \Phi^n f \rangle$ may not converge at all, as evidenced by Example 3 in the last section.[12] This condition however is easy to obtain if there are only two actions and two output levels, with the aid of the following assumption.

A 3. There are two outputs and two actions a_1, a_2 with $g(a_2) > g(a_1)$. In static problem it is strictly optimal for principal to implement action a_2 given zero reservation utility for the agent.

LEMMA 5.1. *Suppose A3 holds. Let* $0 \leq \ell_0 \leq \bar{\xi}$ *and* $f_0 : [\ell_0, \bar{\xi}\,] \rightarrow \Re$ *be continuous, strictly decreasing, and* $f_0(\xi) \leq V^f(\xi)$, $\forall \xi \in [\ell_0, \bar{\xi}\,]$. *For* $n = 1, 2...$, *let* $[\ell_n, \bar{\xi}\,]$ *be the domain of function* $\Phi^n f_0$. *Then there exists* $\ell^* \in [0, \bar{\xi}\,]$ *such that the sequence* $\langle \ell_n \rangle$ *converges to* ℓ^*.

Now Lemma 5.1 and Proposition 2 imply the following main result for the 2 × 2 case.

PROPOSITION 3. *Suppose A3 holds. Then a unique RP value function exists.*

When the convergence result is applicable there is an intuitive interpretation for Self-Pareto-Generating as follows. Call a contract σ one-period RP if it is Pareto optimal. For $T \geq 2$, contract σ is T-period RP if $\forall\, h^1$, continuation contract $\sigma | h^1$ is T-1-period RP and σ is Pareto optimal among all incentive compatible and feasible contracts whose continuation contracts are T-1-period RP. Contract σ is called ∞-*period* RP if it is T-period RP for all T. Let f_T be the value function associated with T-period RP contracts. If the sequence $\langle f_T \rangle$ converges, then the limit function is the value function for ∞-period RP contracts and is also Self-Pareto-Generating. Therefore ∞-period RP and Self-Pareto-Generating coincide.

A Proofs

Lemmas A.1 - A.4 are used to prove Proposition 1. The first two lemmas concern generating operator Γ, defined in (2). The proof of the first is obvious so is omitted.

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[12] The sequence of domains indeed oscillates and does not converge.

Lemma A.1. *Let* $f_1 : [\ell_1, \bar{\xi}\,] \longrightarrow \Re$, *and* $f_2 : [\ell_2, \bar{\xi}\,] \longrightarrow \Re$ *be two continuous functions. Suppose* $0 \le \ell_1 \le \ell_2$, *and* $f_1(\xi) \ge f_2(\xi)$, *for all* $\xi \in [\ell_2, \bar{\xi}]$. *Then* $\Gamma f_1(\xi) \ge \Gamma f_2(\xi)$ *whenever both sides are defined.*

Lemma A.2. *There exist some* $\bar{w} > 0$ *and some* $M > 0$ *such that for any continuous function* $f : [\ell, \bar{\xi}\,] \to \Re$, *the generating operator* Γ *satisfies (i) optimal wage payments are no more than* $\bar{w}$; *(ii) the generated function* $\Gamma(f)$ *is concave, and the left and right derivatives of* $\Gamma(f)$ *are bounded from below by* $-M$.

Proof of Lemma A.2. To prove part (i), let $\underline{p}$ be the $\min\{p_i(a) : \forall a \in A,\ \forall i = 1, ..., I.\}$; let $\bar{g} = \max\{g(a) : \forall a \in A\}$; and let $\bar{w}$ be such that $u(\bar{w})\underline{p} - \bar{g} = \bar{\xi}$. Concavity of $\Gamma(f)$ follows from randomization. To prove the rest of part (ii), using a variation argument one finds from Eq. (2) that the left derivative of $\Gamma(f)$ at $\bar{\xi}$ will be no less than

$$\sum_j \pi^j \sum_i -\frac{p_i(a^j)}{u'(w_i^j)}$$

which together with part (i) implies that the left derivative of $\Gamma(f)$ at $\bar{\xi}$ is no smaller than $-M \equiv -\frac{1}{u'(\bar{w})}$. Since $\Gamma(f)$ is concave, the left and right derivatives of $\Gamma(f)$ at every point must be greater than or equal to $-M$. QED

Lemmas A.3 and A.4 below concern the family of value functions $V(\cdot, \ell)$. Let T_ℓ be the contraction mapping operator embedded in functional equation (5). Then by definition for any continuous function $f : [\ell, \bar{\xi}\,] \to \Re$, the function $T_\ell f(\cdot, \ell)$ is the restriction of the generated function $\Gamma f(\cdot, \ell)$ (whose domain is $[\delta\ell, \bar{\xi}\,]$) to interval $[\ell, \bar{\xi}\,]$. Lemma A.3 illustrates the effect of changing the lower bound ℓ on value function $V(\cdot, \ell)$.

Lemma A.3. *If* $0 \le \ell' < \ell \le \bar{\xi}$, *then* $V(\xi, \ell') \ge V(\xi, \ell)$, $\forall \xi \in [\ell, \bar{\xi}]$ *and* $\Gamma V(\xi, \ell') \ge \Gamma V(\xi, \ell)$ *whenever both sides are defined.*

Proof of Lemma A.3. Let $f : [\ell', \bar{\xi}] \longrightarrow \Re$ and $h : [\ell, \bar{\xi}] \longrightarrow \Re$ be identically zero on their respective domains. Then by Lemma A.1, $\Gamma f(\xi) \ge \Gamma h(\xi)$, $\forall \xi \in [\ell, \bar{\xi}\,]$, hence $T_{\ell'} f(\xi) \ge T_\ell h(\xi)$, $\forall \xi \in [\ell, \bar{\xi}]$. Again by Lemma A.1, $\Gamma(T_{\ell'} f)(\xi) \ge \Gamma(T_\ell h)(\xi)$, so $T_{\ell'}^2 f(\xi) \ge T_\ell^2 h(\xi)$, $\forall \xi \in [\ell, \bar{\xi}\,]$. Keep applying Lemma A.1, one has $\forall n = 1, 2, ...$, $\forall \xi \in [\ell, \bar{\xi}\,]$, $T_{\ell'}^n f(\xi) \ge T_\ell^n h(\xi)$. Note that the two sides converge to $V(\cdot, \ell')$ and $V(\cdot, \ell)$ respectively. Applying Lemma A.1 one more time proves the other half of the Lemma. QED

The next lemma shows that the family of value functions $V(\cdot, \ell)$ has a certain sense of continuity with respect to variable ℓ: as the lower bounds get close, the value functions also get close *uniformly* on the intersection of their domains.

Lemma A.4. *For all* $\ell \in [0, \bar{\xi}\,]$, *it holds that*

$$\lim_{\ell' \to \ell} \left\{ \sup_{\xi \in [\max(\ell, \ell'), \bar{\xi}]} |V(\xi, \ell) - V(\xi, \ell')| \right\} = 0$$

Proof of Lemma A.4. The proof consists of two parts.

Part I. Show left continuity at each $\ell \in (0, \bar{\xi}]$.

The strategy is as follows. Fix $\ell \in (0, \bar{\xi}]$. Let T be the contraction mapping operator defined by functional equation (5) with ℓ being the lower bound on promised utility. For $0 < \ell' < \ell$, let f be the restriction of function $V(\cdot, \ell')$ to $[\ell, \bar{\xi}]$. I will repeatedly apply operator T to function f and show that the distance between f and $T^n f$ for all n, is bounded by a constant multiple of $|\ell - \ell'|$. Since $T^n f$ converges to $V(\cdot, \ell)$, continuity will follow.

I first prove that the left and right derivatives of concave function $V(\cdot, \ell')$ are uniformly bounded for all $\ell' \in [x, \ell)$, where $x \equiv \frac{1}{2}(\delta\ell + \ell)$.

By Lemma A.2, the derivatives of $V(\cdot, \ell')$ are bounded from below by some $-M$, so only need to find an upper bound.

Let $V^f : [0, \bar{\xi}] \longrightarrow \Re$ be the full-commitment value function. Let

$$K' \equiv \frac{V^f(x) - \Gamma V(\delta\ell, \ell)}{x - \delta\ell}.$$

Since by Lemma A.3 $V^f(x) \geq \Gamma V(x, \ell')$ and $\Gamma V(\delta\ell, \ell') \geq \Gamma V(\delta\ell, \ell)$, it follows

$$\frac{\Gamma V(x, \ell') - \Gamma V(\delta\ell, \ell')}{x - \delta\ell} \leq K'$$

Since $\Gamma V(\cdot, \ell')$ is concave, the left and right derivatives of $\Gamma V(\cdot, \ell')$ at any $\xi \in [x, \bar{\xi}]$ are less than K'. Let $K = \max\{M, |K'|\}$. Then for all $\ell' \in [x, \ell)$, the left and right derivatives of $V(\cdot, \ell')$ are within the interval $[-K, K]$.

Next I show that there exists some $B > 0$ such that for all $\xi \in [\ell, \bar{\xi}]$, $f(\xi) \leq Tf(\xi) + B\gamma$ where $\gamma = \ell - \ell'$.

For any $\xi \in [\ell, \bar{\xi}]$, let $(\pi^j, (w_i^j), (\xi_i^j), a^j)_{j \in J}$ be a solution that attains value $V(\xi - \gamma, \ell')$, given continuation value function $V(\cdot, \ell')$, i.e.

$$V(\xi - \gamma, \ell') = \sum_j \pi^j \sum_i p_i(a^j) \left[y_i - u^{-1}(z_i^j) + \delta V(\xi_i^j, \ell') \right], \tag{7}$$

where each $z_i^j \equiv u^{-1}(w_i^j)$, $\forall i$, $\forall j$.

Now for each j, construct vectors $(\tilde{z}_i^j)$, $(\tilde{\xi}_i^j)$ such that for all i,

$$\tilde{z}_i^j \geq z_i^j,\ \tilde{\xi}_i^j \geq \xi_i^j,\ \tilde{\xi}_i \geq \ell, \text{ and } \tilde{z}_i^j + \delta\tilde{\xi}_i = z_i^j + \delta\xi_i^j + \gamma. \tag{8}$$

Clearly, the tuple $(\pi^j, (\tilde{w}_i^j), (\tilde{\xi}_i^j), a^j)_{j \in J}$ is feasible, incentive compatible, and gives the agent expected utility ξ.

By equation (8) it follows

$$-u^{-1}(\tilde{z}_i^j) \geq -u^{-1}(z_i^j + \gamma) \geq -M\gamma - u^{-1}(z_i^j),\ \forall j,\ \forall i \tag{9}$$

where $M > 0$ again is defined in Lemma A.2.

Moreover, since $|\tilde{\xi}_i^j - \xi_i^j| \leq \gamma$ and $V(\cdot, \ell')$ has bounded variation, then

$$V(\tilde{\xi}_i^j, \ell') \geq V(\xi_i^j, \ell') - K\gamma,\ \forall j,\ \forall i. \tag{10}$$

By equations (7), (9), (10) and bounded variation of $V(\cdot, \ell')$, it follows

$$Tf(\xi) \geq V(\xi - \gamma, \ell') - K\gamma \geq V(\xi, \ell') - 2K\gamma, \ \ \forall \xi \in [\ell, \bar{\xi}].$$

Letting $B \equiv 2K$ and noting that f is the restriction of $V(\cdot, \ell')$ to $[\ell, \bar{\xi}]$, one has

$$f(\xi) \leq Tf(\xi) + B\gamma. \tag{11}$$

Since operator T is monotone and discounting, it follows

$$Tf(\xi) \leq T(Tf)(\xi) + \delta B\gamma,$$

which again by (11) implies

$$f(\xi) \leq (T^2 f)(\xi) + B\gamma + \delta B\gamma.$$

Repeatedly applying operator T on both sides and using (11), one obtains, for all n,

$$f(\xi) \leq T^n f(\xi) + B\gamma + \delta B\gamma + ... + \delta^{n-1} B\gamma.$$

Since $\lim_{n\to\infty} T^n f(\xi) = V(\xi, \ell)$ for all $\xi \in [\ell, \bar{\xi}]$, it follows

$$V(\xi, \ell') = f(\xi) \leq \frac{B\gamma}{1-\delta} + V(\xi, \ell).$$

Therefore for all $\xi \in [\ell, \bar{\xi}]$,

$$\left| V(\xi, \ell') - V(\xi, \ell) \right| \leq \frac{B\gamma}{1-\delta} = \frac{B}{1-\delta} |\ell - \ell'|.$$

This establishes left continuity at $\ell \in (0, \bar{\xi}]$.

Part II. Show right continuity at any $\ell \in [0, \bar{\xi})$.

The argument parallels that in Part I. Consider an $\ell' > \ell$. Let f be the restriction of $V(\cdot, \ell)$ on $[\ell', \bar{\xi}]$. Let T be the contraction mapping operator embedded in Eq. (5) when the lower bound on promised utility is ℓ'. The strategy again is to keep applying operator T on f and show that the distance between f and $T^n f$ is bounded by some constant multiple of $|\ell - \ell'|$. The only difference from Part I is the argument for bounded variation of function $V(\cdot, \ell)$, which in fact is much simpler to obtain in this case. A lower bounded $-M$ on the left and right derivatives of $V(\cdot, \ell)$ is already known to exist; concave function $V(\cdot, \ell)$ can be extended to concave function $\Gamma V(\cdot, \ell)$ on $[\delta\ell, \bar{\xi}]$, which implies the existence of finite left derivative at ℓ and that is an upper bound. The proof then goes through as in Part I. QED

Proof of Proposition 1. The proof is by contradiction. Function $V(\cdot, \ell^*)$ is nonincreasing as it is the limit of a sequence of nonincreasing functions. Suppose $V(\cdot, \ell^*)$ is not Semi-RP. Then there must exist some $\hat{\xi} \in [0, \ell^*)$ such that $\Gamma V(\hat{\xi}, \ell^*) > V(\ell^*, \ell^*)$. I will show that then there exists some $\ell' < \ell^*$ such that $\Gamma V(\hat{\xi}, \ell') < \Gamma V(\hat{\xi}, \ell^*)$, which is a contradiction by Lemma A.3. For later reference, let $\Delta \equiv \Gamma V(\hat{\xi}, \ell^*) - V(\ell^*, \ell^*) > 0$.

By Lemma A.4, given any $\epsilon \in (0, \frac{\Delta}{2})$, there exists some $\eta' > 0$ such that

$$\forall \ell' \in (\ell^* - \eta', \ell^*), \quad |V(\ell^*, \ell^*) - V(\ell^*, \ell')| < \epsilon.$$

Since $V(\ell^*, \ell^*) \leq V(\ell^*, \ell')$ by Lemma A.3, it follows that

$$V(\ell^*, \ell^*) + \epsilon > V(\ell^*, \ell'). \tag{12}$$

Now by Lemma A.2, there exists some $M > 0$ such that $\forall \, \ell' \in [0, \bar{\xi}]$ and $\forall \xi \in [\ell', \ell^*]$,

$$V(\ell^*, \ell') + (\ell^* - \xi)M \geq V(\xi, \ell'). \tag{13}$$

Let $\eta = \min\{\eta', \frac{\Delta}{2M}\}$. Then for all $\ell' \in (\ell^* - \eta, \ell^*)$ and $\forall \xi \in [\ell', \bar{\xi}]$,

$$\eta M \geq (\ell^* - \xi)M. \tag{14}$$

Adding up Eq.(14) and Eq. (12), one has

$$V(\ell^*, \ell^*) + \epsilon + \eta M \geq V(\ell^*, \ell') + (\ell^* - \xi)M. \tag{15}$$

Equations(15) and (13) imply that $\forall \ell' \in (\ell^* - \eta, \ell^*)$ and $\forall \xi \in [\ell', \bar{\xi}]$,

$$V(\ell^*, \ell^*) + \epsilon + \eta M \geq V(\xi, \ell'),$$

Since $\epsilon < \Delta/2$ and $\eta < \Delta/(2M)$, it follows

$$V(\ell^*, \ell^*) + \Delta > V(\xi, \ell'). \tag{16}$$

By the definition of ℓ^*, $\forall \, \ell' \in [0, \ell^*)$, $V(\cdot, \ell')$ attains its maximum somewhere in $(\ell', \bar{\xi}\,]$. Concavity of the generated function $\Gamma V(\cdot, \ell')$ and the fact that $V(\cdot, \ell')$ and $\Gamma V(\cdot, \ell')$ coincide on $[\ell', \bar{\xi}\,]$ imply that $\Gamma V(\cdot, \ell')$ also attains its (identical) maximum at the same points.

Thus by Eq.(16) the maximum of $V(\cdot, \ell')$, hence the *maximum* of $\Gamma V(\cdot, \ell')$ will be less than $V(\ell^*, \ell^*) + \Delta = \Gamma V(\hat{\xi}, \ell^*)$. Therefore $\Gamma V(\hat{\xi}, \ell') < \Gamma V(\hat{\xi}, \ell^*)$, which is impossible by Lemma A.3, because $\ell' < \ell^*$. The last sentence of the Proposition again follows from Lemma A.3. QED

The following Lemma is needed for proving Proposition 2.

LEMMA A.5. *For* $n = 0, 1, ...$ *let* $f_n : [\ell_n, \bar{\xi}\,] \to \Re$ *with* $0 \leq \ell_n \leq \bar{\xi}$ *and* $f_n(\xi) \leq V^f(\xi)$ *for all* $\xi \in [\ell_n, \bar{\xi}\,]$*, where* V^f *is the full-commitment value function. If* $\lim_{n\to\infty} d(f_0, f_n) = 0$ *then*

$$\lim_{n\to\infty} d(\Gamma f_0, \Gamma f_n) = 0, \text{and} \lim_{n\to\infty} d(\Phi f_0, \Phi f_n) = 0.$$

Proof of Lemma A.5. I first prove that there exists a constant $K > 0$ such that the left and right derivatives of function Γf_0 are bounded within the interval $[-K, K]$. To begin, note that by Lemma A.2 there exists $M > 0$ such that the left and right derivatives of function Γf_0 are bounded from below by $-M$. To find an upper bound, let V_{max} be the maximum of function Γf_0. Let β be the minimum promised payoff to the agent while the principal is still able to implement an action a with $g(a) > 0$. Note that $\beta > \delta \ell_0$. Define

$K' = (V_{max} - \Gamma f_0(\delta\ell_0))/(\beta - \delta\ell_0)$. Clearly, the left and right derivatives of function Γf_0 are bounded from above by K'. The claim follows by taking $K = \max\{K', M\}$.

Next, for each $n = 0, 1, ...,$ let $\gamma_n \equiv |\ell_n - \ell_0|$ and $d_n \equiv d(f_0, f_n) \geq \gamma_n$. Note that $\gamma_n \to 0$ and $d_n \to 0$. I will show that there exists some $B > 0$ such that for all n, $|\Gamma f_n(\xi) - \Gamma f_0(\xi)|$ is uniformly bounded on the interval $[\max(\delta\ell_n, \delta\ell_0), \bar{\xi}\,]$ by Bd_n. The conclusion of the lemma will then follow from there.

First consider the case when $\ell_n < \ell_0$. Fix $\xi \in [\delta\ell_0, \bar{\xi}\,]$. Let $(\pi^j, (w_i^j), (\xi_i^j), a^j)_{j\in J}$ be a solution that attains value $\Gamma f_0(\xi)$. It follows that

$$\Gamma f_0(\xi) = \sum_{j\in J} \pi^j \sum_i p_i(a^j) \left\{ y_i - w_i^j + \delta f_0(\xi_i^j) \right\}$$

$$\leq \sum_{j\in J} \pi^j \sum_i p_i(a^j) \left\{ y_i - w_i^j + \delta [f_n(\xi_i^j) + d_n] \right\} \leq \Gamma f_n(\xi) + d_n.$$

On the other hand, let $(\pi^k, (w_i^k), (\xi_i^k), a^k)_k$ be a solution that attains $\Gamma f_n(\xi)$. Then

$$\Gamma f_n(\xi) = \sum_k \pi^k \sum_i p_i(a^k) \left\{ y_i - w_i^k + \delta f_n(\xi_i^k) \right\}$$

$$\leq \sum_k \pi^k \sum_i p_i(a^k) \left\{ y_i - w_i^k + \delta \left[f_n(\xi_i^k + \gamma_n) + M\gamma_n \right] \right\}$$

$$\leq \sum_k \pi^k \sum_i p_i(a^k) \left\{ y_i - w_i^k + \delta \left[f_0(\xi_i^k + \gamma_n) + d_n \right] \right\} + M\gamma_n$$

$$\leq \Gamma f_0(\xi + \gamma_n) + (M+1)d_n$$

$$\leq \Gamma f_0(\xi) + K\gamma_n + (M+1)d_n$$

$$\leq \Gamma f_0(\xi) + (K+M+1)d_n.$$

The first inequality follows because the left and right derivatives of f_n are within $[-M, 0]$; the second follows by the definition of d_n; the third by the suboptimality of menu $(\pi^k, (w_i^k), (\xi_i^k + \gamma_n), a^k)_k$ for generating $\Gamma f_0(\xi + \gamma_n)$; the fourth by bounded variation of Γf_0.

The results are similar if $\ell_n > \ell_0$. In conclusion, this proves that there exists some B such that $|\Gamma f_n(\xi) - \Gamma f_0(\xi)| < Bd_n$ for all $\xi \in [\max(\delta\ell_n, \delta\ell_0), \bar{\xi}\,]$. Therefore $d(\Gamma f_n, \Gamma f_0) \to 0$.

Now let a subsequence $\langle \Phi f_{n_k} \rangle$ converge to a function $h : [\ell', \bar{\xi}\,] \longrightarrow \Re$. Let $\Phi f_0 : [\hat{\ell}, \bar{\xi}\,] \longrightarrow \Re$. Note that Φf_0 and h coincide on $[\max(\ell', \hat{\ell}), \bar{\xi}\,]$. If $\hat{\ell} < \ell'$ then

$$\Gamma f_0(\hat{\ell}) = \Phi f_0(\hat{\ell}) > \Phi f_0(\ell') = h(\ell') \geq \lim_{n_k\to\infty} \Gamma f_{n_k}(\hat{\ell}) = \Gamma f_0(\hat{\ell}),$$

which is impossible. Similarly $\hat{\ell} > \ell'$ is impossible. Thus $\hat{\ell} = \ell'$ and $d(\Phi f_0, h) = 0$. QED

Proof of Proposition 2. First, I show that there is a subsequence $\langle \Phi^{n_k} f \rangle$ that converge to some function $f^{**} : [\ell^*, \bar{\xi}] \longrightarrow \Re$. Extend each function $\Phi^n f$ to the entire interval $[0, \bar{\xi}]$ by letting $\Phi^n f(\xi)$ equal to the maximum of $\Phi^n f$ for each ξ that was not previously in its domain. Graphically, this amounts to horizontally extending the graph of $\Phi^n f$ to hit the vertical axis. It is straightforward to verify that the family of the extended functions $\langle \Phi^n f \rangle$

is equicontinuous. Then by Ascoli-Arzela Theorem (see Royden (1988), page 169.), there exists a subsequence $\langle \Phi^{n_k} f \rangle$ that uniformly converge to a continuous function f^{**}. Now the sequence $\langle \Phi^{n_k} f \rangle$ without extension converge to the restriction of f^{**} on $[\ell^*, \bar{\xi}]$, which will still be denoted by f^{**} to lessen notational burden.

Second, I show that there is a subsequence $\langle \Phi^{m_j} f \rangle$ that converge to the fixed point, $f^* : [\ell^*, \bar{\xi}] \longrightarrow \Re$, of the contraction mapping operator T defined in Eq. (5), i.e. $Tf^* = f^*$. Fix an $\epsilon > 0$. Consider the sequence $\langle \Gamma(\Phi^{n_k} f) \rangle$, each term of which is obtained by applying Γ to a term in $\langle \Phi^{n_k} f \rangle$. By Lemma A.5, $\langle \Gamma(\Phi^{n_k} f) \rangle$ converge to Γf^{**}. Now $\langle \Phi^{n_k+1} f \rangle$ converge to Tf^{**}, the restriction of Γf^{**} to $[\ell^*, \bar{\xi}]$. Pick one element of the sequence and denote it by $\Phi^{m_1} f$, so that $d(\Phi^{m_1} f, Tf^{**}) \leq \frac{\epsilon}{2}$. Similarly, $\langle \Phi^{n_k+j} f \rangle$ converge to $T^j f^{**}$. I then pick $\Phi^{m_j} f$, so that $d(\Phi^{m_j} f, T^j f^{**}) \leq \frac{\epsilon}{2^j}$. Since $\langle T^j f^{**} \rangle$ converge to the fixed point $f^* = Tf^*$, the sequence $\langle \Phi^{m_j} f \rangle$ converge to f^*.

Next, I show that $\langle \Phi^n f \rangle$ converge to f^*. Since $\langle \Phi^{m_j} f \rangle$ converge to f^*, it follows $\langle \Phi^{m_j+1} f \rangle$ converge to $Tf^* = f^*$. By induction, $\langle \Phi^{m_j+k} f \rangle$ converges to f^*, for all $k = 1, 2, \cdots$. The union of these sequences, excluding repetitions of terms, differ from $\langle \Phi^n f \rangle$ for only finitely many elements. It follows that $\langle \Phi^n f \rangle$ converge to f^*.

Finally, by Lemma A.5, $\langle \Phi(\Phi^n f) \rangle$ converge to $\Phi(f^*)$, which implies $f^* = \Phi f^*$. The last sentence of the proposition obviously holds. QED

Proof of Lemma 5.1. I first prove that given any nonincreasing continuation value function it is strictly optimal for the principal to implement action a_2. To see this, for $i = 1, 2$, let $p_i = p(y_2|a_i)$ and let $E_i = p_i y_2 + (1 - p_i) y_1$ be the expected output given action a_i. In the static problem, the maximum return the principal receives by implementing a_1 is E_1. The maximum return by implementing a_2 is $E_2 - p_2 w_2^*$, where w_2^* is implicitly given by

$$(p_2 - p_1) u(w_2^*) - g(a_2) = 0.$$

By Assumption A3,

$$E_2 - p_2 w_2^* > E_1. \tag{17}$$

Now given a nonincreasing function $f : [\ell, \bar{\xi}\,] \longrightarrow \Re$, one way to implement action a_2 is to use wage scheme $(w_1^* = 0, w_2^*)$ and promise continuation utility ℓ regardless of output. The return to the principal is:

$$E_2 + \delta f(\ell) - p_2 w_2^*,$$

which by Eq. (17) is greater than $E_1 + \delta f(\ell)$, the maximum return by implementing a_1.

It follows that for $f_n \equiv \Phi^n f_0$, the maximum of function Γf_n is attained by implementing a_2. Let $(z_1, z_2), (\xi_1, \xi_2)$, where $z_1, z_2 \geq 0$ and $\xi_1, \xi_2 \in [\ell_n, \bar{\xi}\,]$, be a vector of current and promised utilities that attain the maximum of Γf_n. It follows that they must satisfy:

$$z_1 = 0; \;\; \xi_1 = \ell_n$$

$$(p_2 - p_1)(z_2 + \delta \xi_2 - z_1 - \delta \xi_1) = g(a_2).$$

Thus the point ℓ_{n+1} is given by

$$\ell_{n+1} = p_2(z_2 + \delta\xi_2) + (1 - p_2)(z_1 + \delta\xi_1) - g(a_2)$$

which by the two preceding equations becomes

$$\ell_{n+1} = \delta\ell_n + \frac{p_1 g(a_2)}{p_2 - p_1}.$$

It follows that $\ell_n = \delta^n \ell_0 + (\delta^{n-1} + ... + \delta + 1)\frac{p_1 g(a_2)}{p_2 - p_1} = \delta^n \ell_0 + (1 - \delta^n)\ell^*$, where $\ell^* \equiv \frac{p_1 g(a_2)}{(1-\delta)(p_2 - p_1)}$. Clearly $\ell_n \longrightarrow \ell^*$. QED

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