Class 2: Compound Interest

Under compound interest, the whole balance in the account earns interest.

$$a(1) = 1 + i,\ a(2) = (1+i) + i(1+i) = (1+i)^2,\ a(3) = (1+i)^2 + i(1+i)^2 = (1+i)^3,\ \ldots$$

the subsequent balances are calculated by adding the balance in the previous period + the interest earned in the account during that period.

So we obtain the formula: $a(t) = (1+i)^t$
and the corresponding future value formula:

$$S = P(1+i)^t$$

This also give us a formula for present values:

$$P = \frac{S}{(1+i)^t}, \quad \text{where } v = \frac{1}{1+i}.$$

v is called the *discount factor*.

Nominal Rates of Interest

Interest may be converted more frequently than a year. We denote by $i^{(m)}$ the *nominal rate* of interest, when interest is converted m times per year.

In this case, the interest per period is $i^{(m)}/m$.
So the accumulating function is $a(t) = (1 + i^{(m)}/m)^{mt}$, where the time t is measured in years. So the formula for future value is:

$$S = P\ \left(1 + \frac{i^{(m)}}{m}\right)^{mt}$$

This also gives us the formula for Present Value:

$$P = \frac{S}{\left(1 + \frac{i^{(m)}}{m}\right)^{mt}}.$$

Problems:

(1) On July 10, 2006, \$7500 will be invested at 8% converted quarterly. Find the balance on Oct 10, 2006.
(2) An education fund is stablished on a child's 12th birthday, by investing \$6580 in an account earning 6% converted monthly. How much is in the fund on the child's 18th birthdate?
(3) How much should be invested today at 7% converted quarterly, in order to meet a 22,650 obligation in 42 months?
(4) a note of 24,890 matures on March 1, 2009. It is sold on Nov 1, 2003, to yield 8% converted monthly. Find the purchase price and the dicount.
(5) Same as above but the note was sold on Oct 6, 2003.

Note: for periods smaller than the interest conversion period, use simple interest with Banker's rule.

Rates of Discount

If the future value of a note one period ahead in time, it's present value is

$$P = \frac{1}{1+v} = 1 - d.$$

Since $P < 1$ we say it was bought at a discount. The discount in that period is d. If we repeat this for several periods, we have compound discount.

Nominal Rate of Discount: $d^{(m)}$ = rate of discount converted m times per year.
Discount per period: $d^{(m)}/m$.

$$P = S\left(1 - \frac{d^{(m)}}{m}\right)^n, \qquad S = \frac{P}{\left(1 - \frac{d^{(m)}}{m}\right)^n}.$$

(1) Find the accumulated value of 2500 after 2 years at 9% rate of discount converted semiannually.
(2) Find the present value of 10000 5 years from now at 7% rate of discount converted quarterly.

Nominal Rates vs Effective Rates:

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m, \qquad 1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^m,$$

and

$$1 - d = \frac{1}{1+i}.$$

(1) If the nominal rate of discount is 9% converted semiannually, find d.
(2) If the nominal rate of discount is 7% converted quarterly, find i.
(3) Find the effective rate of interest equivalent to a nominal rate of interest of 6% converted monthly.
(4) Find the nominal rate of interest converted semiannually equivalent to a nominal rate of discount of 7% converted quarterly.

Force of Interest Let the nominal rate of interest be fix $= \delta$, converted n times per year, then

$$\left(1 + \frac{\delta}{n}\right)^n \to e^\delta$$

This δ is called the *force of interest*. The corresponding accumulation function is $a(t) = e^{\delta t}$. The effective interest is

$$1 + i = e^\delta, \quad \text{or } \delta = \ln(1+i).$$

(1) Find the accumulated value of 1000 invested for ten years if the force of interest is 5%.
(2) Find the effective interest if the force of interest is 5%.
(3) Find the force of interest if the effective interest rate is 5%.

Other forces of interest

$$\delta(t) = \frac{a'(t)}{a(t)} = (\ln a(t))'$$

which gives

$$a(n) = e^{\int_0^n \delta(t)\ dt}.$$

That is

$$S = Pe^{\int_0^n \delta(t)\ dt}.$$

(1) Find an expression for the Present Value if you know the Future Value S.
(2) Find the accumulated value of 1000 at the end of 8 years if $\delta_t = 1/(1+t)$.
(3) Find the accumulated value of 1000 at the end of 5 years if $\delta_t = t/(1+t^2)$.
(4) Find the present value of a note with maturity value of 1000 in 5 years, if the force of interest is $\delta_t = t/(1+t^2)$.

Accumulation Function, Effective Interest

$a(t)$ = accumulation function = future value of \$1.
$A(t)$ = amount function = future value of an original principal P.

Properties:

- $a(0) = 1$
- $a(t)$ is increasing.
- $A(t) = Pa(t)$.

Interest earned from time t_a to t_b: $A(t_b) - A(t_a)$ (ending - beginning)
Interest earned in the nth period: $I_n = A(n) - A(n-1)$.

Effective rate: interest proportionate to the money at the beginning of the period

$$\frac{A(t_b) - A(t_a)}{A(t_a)} = \frac{a(t_b) - a(t_a)}{a(t_a)}$$

Effective rate of interest during the nth period:

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{a(n) - a(n-1)}{a(n-1)}$$

(1) Find i_n for an account earning 7% a) simple interest, b) compound interest.
(2) Find i_1, i_2, i_3 for an acount with $\delta_t = 3t^2/10$.

Homework
Chapter 1: 1, 4–8, 14–17, 19–22, 28, 30, 33, 38, 40, 42, 43, 45–47, 49–51.