Class 2: Compound Interest

Under compound interest, the whole balance in the account earns interest.

\[ a(1) = 1 + i, \quad a(2) = (1 + i) + i(1 + i) = (1 + i)^2, \quad a(3) = (1 + i)^2 + i(1 + i)^2 = (1 + i)^3, \ldots \]

the subsequent balances are calculated by adding the balance in the previous period + the interest earned in the account during that period.

So we obtain the formula: \( a(t) = (1 + i)^t \)

and the corresponding future value formula:

\[ S = P(1 + i)^t \]

This also give us a formula for present values:

\[ P = \frac{S}{(1 + i)^t}, \quad \text{where } v = \frac{1}{1+i}. \]

\( v \) is called the discount factor.

Nominal Rates of Interest

Interest may be converted more frequently than a year. We denote by \( i^{(m)} \) the nominal rate of interest, when interest is converted \( m \) times per year.

In this case, the interest per period is \( i^{(m)}/m \).

So the accumulating function is \( a(t) = (1 + i^{(m)}/m)^{mt} \), where the time \( t \) is measured in years. So the formula for future value is:

\[ S = P \left( 1 + \frac{i^{(m)}}{m} \right)^{mt} \]

This also gives us the formula for Present Value:

\[ P = \frac{S}{\left( 1 + \frac{i^{(m)}}{m} \right)^{mt}}. \]

Problems:

1. On July 10, 2006, $7500 will be invested at 8% converted quarterly. Find the balance on Oct 10, 2006.
2. An education fund is established on a child’s 12th birthday, by investing $6580 in an account earning 6% converted monthly. How much is in the fund on the child’s 18th birthdate?
3. How much should be invested today at 7% converted quarterly, in order to meet a 22,650 obligation in 42 months?
5. Same as above but the note was sold on Oct 6, 2003.

Note: for periods smaller than the interest conversion period, use simple interest with Banker’s rule.
Rates of Discount

If the future value of a note one period ahead in time, it’s present value is

\[ P = \frac{1}{1 + v} = 1 - d. \]

Since \( P < 1 \) we say it was bought at a discount. The discount in that period is \( d. \) If we repeat this for several periods, we have compound discount.

**Nominal Rate of Discount:** \( d^{(m)} = \) rate of discount converted \( m \) times per year.

Discount per period: \( d^{(m)}/m. \)

\[
P = S \left( 1 - \frac{d^{(m)}}{m} \right)^n, \quad S = \frac{P}{\left( 1 - \frac{d^{(m)}}{m} \right)^n}.
\]

1. Find the accumulated value of 2500 after 2 years at 9% rate of discount converted semiannually.
2. Find the present value of 10000 5 years from now at 7% rate of discount converted quarterly.

**Nominal Rates vs Effective Rates:**

\[
1 + i = \left( 1 + \frac{i^{(m)}}{m} \right)^m, \quad 1 - d = \left( 1 - \frac{d^{(m)}}{m} \right)^m,
\]

and

\[ 1 - d = \frac{1}{1 + i}. \]

1. If the nominal rate of discount is 9% converted semiannually, find \( d. \)
2. If the nominal rate of discount is 7% converted quarterly, find \( i. \)
3. Find the effective rate of interest equivalent to a nominal rate of interest of 6% converted monthly.
4. Find the nominal rate of interest converted semiannually equivalent to a nominal rate of interest of 7% converted quarterly.

**Force of Interest** Let the nominal rate of interest be \( f = \delta, \) converted \( n \) times per year, then

\[
\left( 1 + \frac{\delta}{n} \right)^n \to e^\delta
\]

This \( \delta \) is called the force of interest. The corresponding accumulation function is \( a(t) = e^{\delta t}. \) The effective interest is

\[ 1 + i = e^\delta, \quad \text{or} \quad \delta = \ln(1 + i). \]

1. Find the accumulated value of 1000 invested for ten years if the force of interest is 5%.
2. Find the effective interest if the force of interest is 5%.
3. Find the force of interest if the effective interest rate is 5%.

Other forces of interest

\[
\delta(t) = \frac{a'(t)}{a(t)} = (\ln a(t))'
\]

which gives

\[ a(n) = e^{\int_0^n \delta(t) \, dt}. \]
That is

\[ S = Pe^{\int_0^t \delta(t) \, dt}. \]

(1) Find an expression for the Present Value if you know the Future Value \( S \).
(2) Find the accumulated value of 1000 at the end of 8 years if \( \delta_t = 1/(1 + t) \).
(3) Find the accumulated value of 1000 at the end of 5 years if \( \delta_t = t/(1 + t^2) \).
(4) Find the present value of a note with maturity value of 1000 in 5 years, if the force of interest is \( \delta_t = t/(1 + t^2) \).

**Accumulation Function, Effective Interest**

\( a(t) = \) accumulation function = future value of $1.
\( A(t) = \) amount function = future value of an original principal \( P \).

Properties:
- \( a(0) = 1 \)
- \( a(t) \) is increasing.
- \( A(t) = Pa(t) \).

Interest earned from time \( t_a \) to \( t_b \): \( A(t_b) - A(t_a) \) (ending - beginning)
Interest earned in the \( n \)th period: \( I_n = A(n) - A(n - 1) \).

Effective rate: interest proportionate to the money at the beginning of the period

\[ \frac{A(t_b) - A(t_a)}{A(t_a)} = \frac{a(t_b) - a(t_a)}{a(t_a)} \]

Effective rate of interest during the \( n \)th period:

\[ i_n = \frac{A(n) - A(n - 1)}{A(n - 1)} = \frac{a(n) - a(n - 1)}{a(n - 1)} \]

(1) Find \( i_n \) for an account earning 7\% a) simple interest, b) compound interest.
(2) Find \( i_1, i_2, i_3 \) for an account with \( \delta_t = 3t^2/10 \).

**Homework**
Chapter 1: 1, 4–8, 14–17, 19–22, 28, 30, 33, 38, 40, 42, 43, 45–47, 49–51.