Outstanding loan balance
$L = \text{value of the loan at time } t = 0.$
The loan is repaid with $n$ payments of $X_k$ at the end of period $k$.
$B_k = \text{outstanding loan balance at the end of period } k.$

The outstanding loan balance is equal to:
(a) the present value of remaining payments (prospective method)
(b) or the accumulation value of the loan — the accumulation value of previous payments (retrospective method)
Both methods give the same result. In mathematical terms

$$L = B_0 = X_1v + X_2v^2 + \ldots + X_nv^n$$

$$B_k = X_{k+1}v + X_{k+1}v^2 + \ldots + X_nv^{n-k} \text{ prospective method}$$
$$= L(1+i)^k - X_1(1+i)^{k-1} - X_2(1+i)^{k-2} - \ldots - X_k \text{ retrospective method}$$

If the loan is paid with level payments, that is, $X_i = X$ for all $i$, then we have

$$L = Xa_{\overline{m}|i} \text{ and } B_k = Xa_{\overline{n-k}|i} = L(1+i)^k - Xs_{\overline{k}|i}.$$  

Amortization schedules
The loan $L$ is repaid with $n$ payments of $X_k$ at time $k$.
The interest portion of the $k^{th}$ payment is the interest of the outstanding loan balance at time $k - 1$, that is

$$I_k = iB_{k-1}$$

The amount of principal repaid in the $k^{th}$ payment is

$$P_k = X_k - I_k$$

In case the loan is paid with $n$ level payments of $X$, we have

$$I_k = X(1 - v^{n-k+1}) \text{ and } P_k = Xv^{n-k+1}.$$  

Sinking Funds
In this case, the loan $L$ is repaid by a lump sum payment at the end $n$ periods. The borrower accumulates a fund, called the sinking fund, at interest $j$ which will be sufficient to exactly repay the loan at the end of the $n$ periods. And the borrower pays the interest on the loan at the end of each period. The interest on the loan is at a rate $i$.

Let $R = \text{periodic payment at the end of each period}$. This payment is decomposed into an interest portion, $iL$, and a payment $X$ into the sinking fund.

$$R = iL + X$$

Since the sinking fund will accumulate to the loan, we have

$$Xs_{\overline{m}|j} = L$$
And since the loan is the present value of all payments we have

\[ L = Ra_{\overline{m}|i;k,j} = i L a_{\overline{m}|i;k,j} + X a_{\overline{m}|i;k,j}. \]

Using the equation for \( X \) we get \( L = i L a_{\overline{m}|i;k,j} + L \frac{a_{\overline{m}|i;k,j}}{s_{\overline{m}|i}} \). Thus we obtain the relationship between the \( i \) and \( j \)

\[ \frac{1}{a_{\overline{m}|i;k,j}} = i + \frac{1}{s_{\overline{m}|i}} \]

**Varying series of payments**

The loan \( L \) is repaid with \( n \) periodic instalments \( R_1, R_2, \ldots, R_n \). So

\[ L = R_1 v + R_2 v^2 + \ldots + R_n v^n. \]

The periodic payment is decomposed into the interest of the loan plus the deposit into the sinking fund \( X_k \)

\[ R_k = i L + X_k \]

The accumulation of the deposits into the sinking fund is the amount of the loan

\[ L = X_1(1 + j)^{n-1} + X_2(1 + j)^{n-2} + \ldots + X_n = R_1(1 + j)^{n-1} + R_2(1 + j)^{n-2} + \ldots + R_n - i L s_{\overline{m}|j} \]

From here we get

\[ L = \frac{R_1(1 + j)^{n-1} + R_2(1 + j)^{n-2} + \ldots + R_n}{1 + i a_{\overline{m}|j}} = \frac{R_1(1/(1+j)) + R_2(1/(1+j))^2 + \ldots + R_n(1/(1+j))^n}{1 + (i-j)a_{\overline{m}|j}} \]