Outstanding loan balance
L = value of the loan at time $t = 0$.
The loan is repaid with n payments of X_k at the end of period k.
B_k = outstanding loan balance at the end of period k.

The **outstanding loan balance** is equal to:
(a) the present value of remaining payments (prospective method)
(b) or the accumulation value of the loan − the accumulation value of previous payments (retrospective method)
Both methods give the same result. In mathematical terms

$$L = B_0 = X_1 v + X_2 v^2 + \ldots + X_n v^n$$

$$\begin{aligned} B_k &= X_{k+1} v + X_{k+1} v^2 + \ldots + X_n v^{n-k} \quad \text{prospective method} \\ &= L(1+i)^k - X_1(1+i)^{k-1} - X_2(1+i)^{k-2} - \ldots - X_k \quad \text{retrospective method} \end{aligned}$$

If the loan is paid with level payments, that is, $X_i = X$ for all i, then we have

$$L = X a_{\overline{n}|i} \quad \text{and} \quad B_k = X a_{\overline{n-k}|i} = L(1+i)^k - X s_{\overline{k}|i}.$$

Amortization schedules
The loan L is repaid with n payments of X_k at time k.
The interest portion of the kth payment is the interest of the outstanding loan balance at time $k-1$, that is

$$I_k = i B_{k-1}$$

The amount of principal repaid in the kth payment is

$$P_k = X_k - I_k$$

In case the loan is paid with n level payments of X, we have

$$I_k = X(1 - v^{n-k+1}) \quad \text{and} \quad P_k = X v^{n-k+1}.$$

Sinking Funds
In this case, the loan L is repaid by a lump sum payment at the end n periods. The borrower accumulates a fund, called the sinking fund, at interest j which will be sufficient to exactly repay the loan at the end of the n periods. And the borrower pays the interest on the loan at the end of each period. The interest on the loan is at a rate i.

Let R = periodic payment at the end of each period. This payment is decomposed into an interest portion, iL, and a payment X into the sinking fund.

$$R = iL + X$$

Since the sinking fund will accumulate to the loan, we have

$$X s_{\overline{n}|j} = L$$

And since the loan is the present value of all payments we have

$$L = Ra_{\overline{n}|i\&j} = iLa_{\overline{n}|i\&j} + Xa_{\overline{n}|i\&j}.$$

Using the equation for X we get $L = iLa_{\overline{n}|i\&j} + L\frac{a_{\overline{n}|i\&j}}{s_{\overline{n}|j}}$. Thus we obtain the relationship between the i and j

$$\frac{1}{a_{\overline{n}|i\&j}} = i + \frac{1}{s_{\overline{n}|i}}$$

Varying series of payments

The loan L is repaid with n periodic instalments $R_1, R_2, \ldots, R_n$. So

$$L = R_1 v + R_2 v^2 + \ldots + R_n v^n.$$

The periodic payment is decomposed into the interest of the loan plus the deposit into the sinking fund X_k

$$R_k = iL + X_k$$

The accumulation of the deposits into the sinking fund is the amount of the loan

$$L = X_1(1+j)^{n-1} + X_2(1+j)^{n-2} + \ldots + X_n = R_1(1+j)^{n-1} + R_2(1+j)^{n-2} + \ldots + R_n - iLs_{\overline{n}|j}$$

From here we get

$$\begin{aligned} L &= \frac{R_1(1+j)^{n-1} + R_2(1+j)^{n-2} + \ldots + R_n}{1 + is_{\overline{n}|j}} \\ &= \frac{R_1(1/(1+j)) + R_2(1/(1+j))^2 + \ldots + R_n(1/(1+j))^n}{1 + (i-j)a_{\overline{n}|j}} \end{aligned}$$