Math 220 - Exam III - Summer 2004

NAME:

Problem 1. (15 pts) Let

\[
T = \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
\end{bmatrix} = \begin{bmatrix}
  x_1 + x_2 - x_3 \\
  x_2 + x_3 - x_4 \\
  x_1 + 2x_2 - x_4 \\
\end{bmatrix}
\]

(a) Find the matrix \( A \) corresponding to \( T \).
(b) Find a basis for \( \text{Null}(A) \).
(c) Find a basis for \( \text{Im}(A) \).
(d) What are the dimensions of \( \text{Null}(A) \) and \( \text{Im}(A) \)?

Problem 2. (15 pts) Determine whether \( U \) is a subspace:

(a) \( U = \{ [s, 0, t, 0] : s, t \in \mathbb{R} \} \).
(b) \( U = \{ [s, 2, t, 0] : s, t \in \mathbb{R} \} \).
(c) \( U = \{ [st, 0, t, 0] : s, t \in \mathbb{R} \} \).

If it is a subspace, check the three properties of subspaces. If it is not a subspace, explain which property fails.

Problem 3. (5 pts) Explain what it means that \( k \) vectors are linearly independent.

Problem 4. (15 pts) Let \( V = [1, 0, -1, 3] \), \( W = [1, 1, 0, 7] \) and \( Z = [1, 1, 0, 3] \).

(a) Show \( V, W, Z \) are linearly independent.
(b) Decide whether \( V, W, Z \) span \( \mathbb{R}^4 \). If yes, explain your answer, if not, find a basis of \( \mathbb{R}^4 \) that contains these vectors.

Problem 5. (10 pts) Show that if \( X, Y \) are independent, then \( 2X - Y \) and \( X + 3Y \) are also independent.

Problem 6. (10 pts) Find a basis for

\[
U = \text{span}\{[-1, 2, 1, 0, 4], [3, 1, 0, -2, 5], [-7, 7, -2, 6, 13], [-2, 3, -1, 2, 9]\}
\]

using the row method.

Problem 7. (15 pts)

(a) Define the linear span of vectors \( v_1, v_2, \ldots, v_k \).
(b) Can 3 vectors span \( \mathbb{R}^4 \)? Explain.
(c) Do 5 vectors always span \( \mathbb{R}^5 \)? Explain.
(d) Find 3 vectors that span \( \mathbb{R}^3 \). Explain.
(e) What is the dimension of the subspace \( U = \{0\} \)?
(f) List all the subspaces of \( \mathbb{R}^2 \).

Problem 8. (15 pts) Use the Gram–Schmidt Algorithm to convert the given basis of a subspace \( U \) into an orthonormal basis \( \{F_1, F_2, F_3\} \) for \( U \).

\[
\{[1, 1, 0, 1], [1, 0, 1, 0], [1, -1, -1, 1]\}
\]