1. (10 pts) Let $T : P_2 \rightarrow \mathbb{R}^3$ be defined by

$$T(a + bx + cx^2) = \begin{bmatrix} a + 2b \\ a + b - c \\ c + 2a \end{bmatrix}$$

a. Let $\beta_1$ = standard basis of $P_2$, and $\beta_2$ = standard basis of $\mathbb{R}^3$. Find $[T]_{\beta_1,\beta_2}$.

b. Decide whether $T$ is an isomorphism. Give an explanation for your answer.
2. (15 pts) Let $T : P_3 \to P_2$ be defined by $T(a + bx + cx^2 + dx^3) = (a + b) + (c - a)x + (b + c + d)x^2$.
   a. Let $S_3$ and $S_2$ be the standard basis of $P_3$ and $P_2$ respectively. Find $[T]_{S_3S_2}$.
   b. Let $\beta = \{-x + x^2, 1 + x, x\}$, and $\beta' = \{-x + x^3, 1 + x^2, x, -1 + x\}$. Find $[T]_{\beta'\beta}$. 
3. (10 pts) Use the definition of determinant (cofactor expansion) to find the determinant of the following matrix:

\[
A = \begin{bmatrix}
2 & 0 & 0 & 0 \\
4 & 0 & 1 & 0 \\
5 & 1 & 2 & 0 \\
7 & 0 & 3 & 1 \\
-8 & 1 & 4 & 1
\end{bmatrix}
\]
4. (10 pts) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ be a matrix with determinant $-3$. Find the determinant of the following matrices:

a. $B = \begin{bmatrix} a_3 & a_1 & a_2 \\ b_3 & b_1 & b_2 \\ c_3 & c_1 & c_2 \end{bmatrix}$

b. $C = \begin{bmatrix} 10b_1 & 10b_2 & 10b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

c. $A^2, AA^T, A^{-3}$.

d. $D = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 - 3a_1 & b_2 - 3a_2 & b_3 - 3a_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

e. $E = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \end{bmatrix}$
5. (25 pts) A group of people buys cars every four years from one of three automobile manufactures, A, B and C. The transition probability of switching from one manufacturer to another is given by the matrix:

$$P = \begin{bmatrix} .5 & .4 & .6 \\ .3 & .4 & .3 \\ 2 & .2 & .1 \end{bmatrix}$$

a. Find the characteristic polynomial $p(\lambda)$, and the eigenvalues.
b. Find the eigenvectors of $P$. (Note: maple doesn’t do a good job for this matrix. Do it yourself by hand.)
c. Diagonalize $P$, that is, find an invertible matrix $Q$ and a diagonal matrix $D$, such that $D = Q^{-1}PQ$.
d. Compute $D^2$, $D^3$. Find an expression for $D^n$ and then find $\lim_{n\to\infty} D^n$.
e. From (c) we have $P = QDQ^{-1}$. Use this expression to find $P^2$ in terms of $D^2$, $P^3$ in terms of $D^3$. Find an expression for $P^n$ in terms of $D^n$. Then find $\lim_{n\to\infty} P^n$.
f. Will one of the manufacturers eventually dominate the market, no mater what the initial sales?