Problem 1. Consider the basis \( v_1 = [1, 1, 0], v_2 = [1, -1, 1], v_3 = [1, -1, -2] \) of \( \mathbb{R}^3 \).

a) Show that the basis is orthogonal, that is: \( v_i \cdot v_j = 0 \) if \( i \neq j \).

b) Let \( B = \{v_1, v_2, v_3\} \) Deduce that \([X]_B = (a, b, c)\), where:

\[
a = \frac{X \cdot v_1}{v_1 \cdot v_1}, \quad b = \frac{X \cdot v_2}{v_2 \cdot v_2}, \quad c = \frac{X \cdot v_3}{v_3 \cdot v_3}
\]

\( \beta = \{v_1, v_2, \ldots, v_k\} \) is an orthogonal basis if \( v_i \cdot v_j = 0 \) for all \( i \neq j \). The basis is called orthonormal, if it is orthogonal and \( \|v_i\| = 1 \) for \( i = 1, \ldots, k \). (That is, all the vectors in the basis are unitary vectors).

It \( \beta \) is orthonormal, and \( u \) is any vector, then

\[
u = (u \cdot v_1)v_1 + (u \cdot v_2)v_2 + \ldots + (u \cdot v_k)v_k.
\]

Problem 2: (a) Show that \( B = \{v_1 = (1, 2, 1, 1), v_2 = (1, 2, -5, 0), v_3 = (-2, 1, 0, 0)\} \) is an orthogonal set in \( \mathbb{R}^4 \).

(b) Let \( S = \text{span}\{v_1, v_2, v_3\} \). Find an orthonormal basis \( \beta \) for \( S \).

(c) Let \( u = (3, 4, 1, 0, -1) \). Find \([u]_\beta\).

MORE ON PROJECTIONS:

If \( V \) is a subspace with orthogonal basis \( B = \{v_1, v_2, \ldots, v_k\} \), the (orthogonal) projection of a vector \( w \) onto \( V \) is given by:

\[
\text{proj}(w, V) = \frac{(w, v_1)}{|v_1|^2}v_1 + \frac{(w, v_2)}{|v_2|^2}v_2 + \cdots + \frac{(w, v_n)}{|v_n|^2}v_n.
\]

Problem 3. Find the projection of the vector \( w = (1, 1, 1, 1) \) onto the subspace \( S \) of \( \mathbb{R}^4 \) generated by the orthogonal basis \( B = \{v_1 = (1, 2, 1, 1), v_2 = (1, 2, -5, 0), v_3 = (-2, 1, 0, 0)\} \)

Problem 4. Let \( S \) be the subspace generated by the vectors \( (1, 1, 1) \) and \( (2, 1, -1) \).

(a) Find an orthogonal basis for \( S \):

(b) Find an orthonormal basis for \( S \):
Gram–Schmidt orthogonalization process:

You have just done a simple example of the Gram–Schmidt process. In general, if one wants to find and orthogonal basis of a space generated by vectors $v_1, v_2, \ldots, v_k$, one does the following iterative scheme:

1. $w_1 = v_1$.
2. Let $S_1 = \text{span}(w_1)$. To get the next vector in the basis, get $v_2$ and substruct the part that is not orthogonal to $w_1$, that is, its projection onto $S_1$:
   \[ w_2 = v_2 - \text{proj}(v_2, w_1) = v_2 - \frac{(v_2, w_1)}{|w_1|^2} w_1. \]
3. Now keep going. Let $S_2 = \text{span}(w_1, w_2)$. To get the next vector in the basis, get $v_3$ and substruct the part that is not orthogonal to $w_1$ and $w_2$, that is, its projection onto $S_2$:
   \[ w_3 = v_3 - \text{proj}(v_3, S_2) = v_3 - \text{proj}(v_3, w_1) - \text{proj}(v_3, w_2) = v_3 - \frac{(v_3, w_1)}{|w_1|^2} w_1 - \frac{(v_3, w_2)}{|w_2|^2} w_2. \]

Continuing in this way, the general expression of the $j$th vector in your basis is:

\[ w_j = v_j - \frac{(v_j, w_1)}{|w_1|^2} w_1 - \frac{(v_j, w_2)}{|w_2|^2} w_2 - \cdots - \frac{(v_j, w_{j-1})}{|w_{j-1}|^2} w_{j-1}. \]

**Problem 5.** (a) Find an orthonormal basis for the subspace $S$ of $R^5$ spanned by the vectors $u_1 = (1, 3, 2, 5, 2), u_2 = (2, 1, 3, 2, 2)$ and $u_3 = (1, 2, 3, 2, 1)$.

(b) Find an orthonormal basis for the subspace $S^k = \{v \in R^5 : v \cdot w = 0 \text{ for any } w \in S\}$.

(c) Let $u = (1, 1, 1, 1, 1)$. Find the vector in $S$ that best approximates $u$. (This vector is the orthogonal projection of $u$ onto $S$).

(d) Let $u = (1, 1, 1, 1, 1)$, find the orthogonal decomposition of $u = u_1 + u_2$ where $u_1 \in S$ and $u_2 \in S^k$.

A square matrix is **orthogonal** if its columns are orthonormal. It turns out $A$ is orthogonal if an only if $A^T A = I$, or $A^{-1} = A^T$.

**Problem 6.** Show that the rotation matrix $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ is orthogonal.

**Symmetric Matrices:** $A$ is symmetric if $A^T = A$.

Symmetric matrices have nice properties:

1. they only have real eigenvalues.
2. any two eigenvectors of a symmetric matrix that correspond to different eigenvalues are orthogonal. That is the eigenspaces are orthogonal.
3. symmetric matrices can be diagonalized using an orthogonal matrix. If $A$ is symmetric then there exists an orthogonal matrix $O$ and a diagonal matrix $D$ such that $O^T A O = D$, where $O$ is the change of basis from an orthogonal basis of eigenvectors to the standard basis.

**Problem 7.** Find the orthogonal diagonalization of $A = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}$.

Read Sections 8.1, 8.2 and 8.5. Do problems Sec. 8.1: 1, 7, 11, 13, 17, 18. Sec. 8.2: 5, 7, 9, 13, 15. Sec. 8.5: 6, 7, 11, 13.