Problem 1. Determinant of elementary matrices: Find the determinants of the 3 by 3 matrices corresponding to the elementary row operations:
(a) swap row 1 and 2, (b) replace row 2 by row $2 - 10$ row 1, (c) divide row 3 by 10.

Note: Since $\text{rref}(A)$ can be obtained from $A$ by using elementary row operations, it turns out $|\text{rref}(A)| = (-1)^{\text{number of swaps (rescaling factors)}}|A|$. In particular $|A| = 0$ if and only if $|\text{rref}(A)| = 0$.

Problem 2. Use the basic properties of determinants to compute the determinant of the matrix
\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 1 & 3 & 1 \\
1 & 1 & 1 & 4
\end{bmatrix}
\]

Problem 3. Find all the values of $s$ and $t$ for which the matrix $A = \begin{bmatrix}
1 & -1 & 1 \\
-2 & s & 3 \\
t & 4 & 5
\end{bmatrix}$ has an inverse.

Problem 4. Let $A = \begin{bmatrix}
1 & 2 & s \\
2 & 3 & t \\
4 & 5 & 7
\end{bmatrix}$, and $B = \begin{bmatrix}
4 & 5 & 8 \\
s & 2 & 3 \\
t & 1 & 8
\end{bmatrix}$. Find all values of $s$ and $t$ for which neither $A$ nor $B$ have an inverse.
Applications:

To find the equation of the line in $\mathbb{R}^2$ through the points $(1,-1), (3,2)$, is equivalent to finding a non-trivial solution $(a, b, c)$ such that

\[
\begin{aligned}
ax + by + c &= 0 \\
 a - b + c &= 0 \\
3a + 2b + c &= 0
\end{aligned}
\]

The only way this system has a non trivial solution is if the coefficients matrix is not invertible. Therefore, the equation of the line is:

\[
\det \begin{bmatrix} x & y & 1 \\ 1 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix} = 0
\]

Which gives: $-3x + 2y + 5 = 0$.

In general, the equation of the line through two points $(x_1, y_1)$ and $(x_2, y_2)$ is

\[
\det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0
\]

To find the equation of the plane in $\mathbb{R}^3$ through the points $(1,-1,2), (3,2,-1), (-4,2,1)$ is equivalent to finding a non-trivial solution $(a, b, c, d)$ such that

\[
\begin{aligned}
ax + by + cz + d &= 0 \\
 a - b + 2c + d &= 0 \\
3a + 2y - c + d &= 0 \\
-4a + 2y + c + d &= 0
\end{aligned}
\]

The only way this system has a non trivial solution is if the coefficients matrix is not invertible. Therefore, the equation of the plane is:

\[
\det \begin{bmatrix} x & y & z & 1 \\ 1 & -1 & 2 & 1 \\ 3 & 2 & -1 & 1 \\ -4 & 2 & 1 & 1 \end{bmatrix} = 0
\]

Which gives: $6x + 17y + 21z - 31 = 0$.

In general, the equation of the plane through three points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and $(x_3, y_3, z_3)$ is

\[
\det \begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0
\]

To find the equation of the parabola of the form $ay + bx^2 + cx + d = 0$ passing through the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, is equivalent to finding a non-trivial solution $(a, b, c, d)$ such that

\[
\begin{aligned}
ay + bx^2 + cx + d &= 0 \\
 ay_1 + bx_1^2 + cx_1 + d &= 0 \\
 ay_2 + bx_2^2 + cx_2 + d &= 0 \\
 ay_3 + bx_3^2 + cx_3 + d &= 0
\end{aligned}
\]
The only way this system has a non trivial solution is if the coefficients matrix is not invertible. Therefore, the equation of the parabola is:

\[
\begin{vmatrix}
  y & x^2 & x & 1 \\
  y_1 & x_1^2 & x_1 & 1 \\
  y_2 & x_2^2 & x_2 & 1 \\
  y_3 & x_3^2 & x_3 & 1 \\
\end{vmatrix}
\]

The equation of the parabola passing through the points (1,0), (2,1), (0,3) is

\[
\begin{vmatrix}
y & x^2 & x & 1 \\
0 & 1 & 1 & 1 \\
1 & 4 & 2 & 1 \\
3 & 0 & 0 & 1 \\
\end{vmatrix}
= 0
\]

Which gives: \(-2y + 4x^2 - 10 + 6\).

To find the equation of a circle through 3 points \((x_1, y_1), (x_2, y_2), (x_3, y_3)\): The equation of the circle is: \((x-a)^2 + (y-b)^2 = R\) can be re-written as: \(A(x^2 + y^2) + Bx + Cy + D = 0\). So all we need to do is find a non-trivial solution to

\[
\begin{cases}
A(x^2 + y^2) + Bx + Cy + D = 0 \\
A(x_1^2 + y_1^2) + Bx_1 + Cy_1 + D = 0 \\
A(x_2^2 + y_2^2) + Bx_2 + Cy_2 + D = 0 \\
A(x_3^2 + y_3^2) + Bx_3 + Cy_3 + D = 0 \\
\end{cases}
\]

The only way this system has a non trivial solution is if the coefficients matrix is not invertible. Therefore, the equation of the circle is:

\[
\begin{vmatrix}
x^2 + y^2 & Bx & Cy & D \\
x_1^2 + y_1^2 & Bx_1 & Cy_1 & D \\
x_2^2 + y_2^2 & Bx_2 & Cy_2 & D \\
x_3^2 + y_3^2 & Bx_3 & Cy_3 & D \\
\end{vmatrix}
= 0
\]

**Problem:** Do problems Sec 6.5: 4, 6, 8, 12.
CRAMER’S RULE

The solution to the system $AX = B$, where $A$ is an invertible matrix, and $X^t = [x_1, x_2, \ldots, x_n]$, is given by the formula:

$$x_i = \frac{\det(\text{replace the } i\text{th. column of } A \text{ by } Y)}{\det(A)}$$

For example, to find the solution to

$$\begin{cases}
x + y - z & = 2 \\
x - y + z & = 3 \\
-x + y + z & = 4
\end{cases}$$

We have

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

So

$$x = \frac{\det \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}} = \frac{-10}{-4} = 2.5$$

$$y = \frac{\det \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ -1 & 4 & 1 \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}} = \frac{-12}{-4} = 3$$

$$z = \frac{\det \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ -1 & 1 & 4 \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}} = \frac{-14}{-4} = 3.5$$

**Problem.** Use Cramer’s rule to find the solution to the system:

$$\begin{cases}
4x + 5y + 3z + 3w & = 1 \\
2x + y + z + w & = 0 \\
2x + 3y + z + w & = 1 \\
5x + 7y + 3z + 4w & = 2
\end{cases}$$

(a) Find $A$ and $B$ such that the system can be written as $AX = B$.

(b) Solve the system using Cramer’s rule.
Another method to find inverses.
Let $A$ be an $n \times n$ matrix. The cofactor coefficient $c_{i,j}$ is $(-1)^{i+j} \ M_{i,j}$. (Recall $M_{i,j}$ = determinant of the matrix $A_{i,j}$ obtained from $A$ by removing the $i$th row and $j$th column). The cofactor matrix $C$ is the matrix whose $(i, j)$ entry is $c_{i,j}$.

The Adjoint matrix to $A$, $\text{Adj}(A) = C^t$, that is, its $(i, j)$ entry is $c_{j,i}$. The inverse of $A$ is $A^{-1} = \text{Adj}(A)/\det(A)$.

Let $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -4 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

(a) The cofactor matrix is:

$$C = \begin{bmatrix} + & -4 & 2 \\ 1 & 2 \\ -3 & 1 & 1 \\ 1 & 2 \\ + & -4 & 2 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 \\ 0 & 1 \\ 1 & 2 \\ 2 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -10 & -4 & 2 \\ 7 & 2 & -1 \\ -2 & 0 & 2 \end{bmatrix}$$

(b) The adjoint matrix to $A$ is

$$\text{Adj}(A) = C^t = \begin{bmatrix} -10 & 7 & -2 \\ -4 & 2 & 0 \\ 2 & -1 & 2 \end{bmatrix}$$

(c) The inverse of $A$ is:

$$\frac{1}{4} \begin{bmatrix} -10 & 7 & -2 \\ -4 & 2 & 0 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -10/4 & 7/4 & -1/2 \\ -1 & 1/2 & 0 \\ 1/2 & -1/4 & 1/2 \end{bmatrix}$$

Problem. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

(a) Find the cofactor matrix.

(b) Find the adjoint matrix to $A$.

(c) Find the inverse of $A$. 
(d) Experiment with the maple command adjoint(A).

HW: Read Section 6.3 and 6.5.