Math 220 - More on Determinants - Fall 2000

Problem 1. Determinat of elementary matrices: Find the determinants of the 3 by 3 matrices corresponding to the elemntary row operations:
(a) swap row 1 and 2, (b) replace row 2 by row 2 − 10 row 1, (c) divide row 3 by 10.

Note: Since rref(A) can be obtained from A by using elementary row operations, it turns out $|\text{rref}(A)| = (-1)^{\text{number of swaps}}(\text{rescaling factors})|A|$. In particular $|A| = 0$ if and only if $|\text{rref}(A)| = 0$.

Problem 2. Use the basic properties of determinants to compute the determinat of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$

Problem 3. Find all the values of s and t for which the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ -2 & s & 3 \\ t & 4 & 5 \end{bmatrix}$ has an inverse.

Problem 4. Let $A = \begin{bmatrix} 1 & 2 & s \\ 2 & 3 & t \\ 4 & 5 & 7 \end{bmatrix}$, and $B = \begin{bmatrix} 4 & 5 & 8 \\ s & 2 & 3 \\ t & 1 & 8 \end{bmatrix}$.
Find all values of s and t for which neither A nor B have an inverse.

Applications:

To find the **equation of the line** in $\mathbb{R}^2$ through the points (1,-1), (3,2), is equivalent to finding a non–trivial solution (a, b, c) such that

$$\begin{cases} ax + by + c = 0 \\ a - b + c = 0 \\ 3a + 2y + c = 0 \end{cases}$$

The only way this system has a non trivial solution is if the coefficients matrix is not invertible. Therefore, the equation of the line is:

$$\det \begin{bmatrix} x & y & 1 \\ 1 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix} = 0$$

Which gives: $-3x + 2y + 5 = 0$.

In general, the equation of the line through two points (x_1, y_1) and (x_2, y_2) is

$$\det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$$

To find the **equation of the plane** in $\mathbb{R}^3$ through the points $(1, -1, 2)$, $(3, 2, -1)$, $(-4, 2, 1)$ is equivalent to finding a non–trivial solution (a, b, c, d) such that

$$\begin{cases} ax + by + cz + d = 0 \\ a - b + 2c + d = 0 \\ 3a + 2y - c + d = 0 \\ -4a + 2y + c + d = 0 \end{cases}$$

The only way this system has a non trivial solution is if the coefficients matrix is not invertible. Therefore, the equation of the plane is:

$$\det \begin{bmatrix} x & y & z & 1 \\ 1 & -1 & 2 & 1 \\ 3 & 2 & -1 & 1 \\ -4 & 2 & 1 & 1 \end{bmatrix} = 0$$

Which gives: $6x + 17y + 21z - 31 = 0$

In general, the equation of the plane through three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\det \begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0$$

To find the **equation of the parabola** of the form $ay + bx^2 + cx + d = 0$ passing through the points (x_1, y_1), (x_2, y_2), (x_3, y_3), is equivalent to finding a non–trivial solution (a, b, c, d) such that

$$\begin{cases} ay + bx^2 + cx + d = 0 \\ ay_1 + bx_1^2 + cx_1 + d = 0 \\ ay_2 + bx_2^2 + cx_2 + d = 0 \\ ay_3 + bx_3^2 + cx_3 + d = 0 \end{cases}$$

The only way this system has a non trivial solution is if the coefficients matrix is not invertible. Therefore, the equation of the parabola is:

$$\det \begin{bmatrix} y & x^2 & x & 1 \\ y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ y_3 & x_3^2 & x_3 & 1 \end{bmatrix}$$

The the equation of the parabola passing through the points (1,0), (2,1), (0,3) is

$$\det \begin{bmatrix} y & x^2 & x & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 4 & 2 & 1 \\ 3 & 0 & 0 & 1 \end{bmatrix} = 0$$

Which gives: $-2y + 4x^2 - 10 + 6$.

To find the equation of a circle through 3 points (x_1, y_1), (x_2, y_2), (x_3, y_3): The equation of the circle is: $(x-a)^2 + (y-b)^2 = R$ cna be re–written as: $A(x^2+y^2) + Bx + Cy + D = 0$. So all we need to do is find a non–trivial solution to

$$\begin{cases} A(x^2+y^2) + Bx + Cy + D = 0 \\ A(x_1^2+y_1^2) + Bx_1 + Cy_1 + D = 0 \\ A(x_2^2+y_2^2) + Bx_2 + Cy_2 + D = 0 \\ A(x_3^2+y_3^2) + Bx_3 + Cy_3 + D = 0 \end{cases}$$

The only way this system has a non trivial solution is if the coefficients matrix is not invertible. Therefore, the equation of the circle is:

$$\det \begin{bmatrix} x^2+y^2 & Bx & Cy & D \\ x_1^2+y_1^2 & Bx_1 & Cy_1 & D \\ x_2^2+y_2^2 & Bx_2 & Cy_2 & D \\ x_3^2+y_3^2 & Bx_3 & Cy_3 & D \end{bmatrix} = 0$$

Problem: Do problems Sec 6.5: 4, 6, 8, 12.

CRAMER'S RULE

The solution to the system $A * X = B$, where A is an invertible matrix, and $X^t = [x_1, x_2, \ldots, x_n]$, is given by the formula:

$$x_i = \frac{\det(\text{ replace the } i\text{th. column of } A \text{ by } Y)}{\det(A)}$$

For example, to find the solution to

$$\begin{cases} x + y - z & = 2 \\ x - y + z & = 3 \\ -x + y + z & = 4 \end{cases}$$

We have

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

So

$$x = \frac{\det \begin{bmatrix} 2 & 1 & -1 \\ 3 & -1 & 1 \\ 4 & 1 & 1 \end{bmatrix}}{\det = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}} = \frac{-10}{-4}, \quad y = \frac{\det \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ -1 & 4 & 1 \end{bmatrix}}{\det = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}} = \frac{-12}{-4}$$

$$z = \frac{\det \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ -1 & 1 & 4 \end{bmatrix}}{\det = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}} = \frac{-14}{-4}$$

Problem. Use Cramer's rule to find te solution to the system:

$$\begin{cases} 4x + 5y + 3z + 3w & = 1 \\ 2x + y + z + w & = 0 \\ 2x + 3y + z + w & = 1 \\ 5x + 7y + 3z + 4w & = 2 \end{cases}$$

(a) Find A abd B such that the system can be written as $AX = B$.

(B) Solve the system using Cramer's rule.

Another method to find inverses.
Let A be and $n \times n$ matrix. The cofactor coeficient $c_{i,j}$ is $= (-1)^{i+j}\ M_{i,j}$. (Recall $M_{i,j}$ = determinant of the matrix $A_{i,j}$ obtained from A by removing the ith row and jth column). The cofactor matrix C is the matrix whose (i,j) entry is $c_{i,j}$.

The Adjoint matrix to A, $\mathrm{Adj}(A) = C^t$, that is, its (i,j) entry is $c_{j,i}$. The inverse of A is $A^{-1} = \mathrm{Adj}(A)/det(A)$.

Let $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -4 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

(a) The cofactor matrix is:

$$C = \begin{bmatrix} +\begin{vmatrix} -4 & 2 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & -4 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix} \\ +\begin{vmatrix} -3 & 1 \\ -4 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -10 & -4 & 2 \\ 7 & 2 & -1 \\ -2 & 0 & 2 \end{bmatrix}$$

(b) The adjoint matrix to A is

$$\mathrm{Adj}(A) = C^t = \begin{bmatrix} -10 & 7 & -2 \\ -4 & 2 & 0 \\ 2 & -1 & 2 \end{bmatrix}$$

(c) The inverse of A is:

$$\frac{1}{4}\begin{bmatrix} -10 & 7 & -2 \\ -4 & 2 & 0 \\ 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -10/4 & 7/4 & -1/2 \\ -1 & 1/2 & 0 \\ 1/2 & -1/4 & 1/2 \end{bmatrix}$$

Problem. Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 3 & -1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$.

(a) Find the cofactor matrix.

(b) Find the adjoint matrix to A.

(c) Find the inverse of A.

(d) Experiment with the maple command adjoint(A).

HW: Read Section 6.3 and 6.5.