Math 220 - Determinants - Fall 2000

The determinant of an $n \times n$ matrix $A$ is:

$$|A| = \text{det}(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \quad \text{det}(A_{ij}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \quad M_{i,j}$$

where $A_{ij}$ is the matrix obtained by removing the $i$th row and the $j$th column. The determinant $\text{det}(A_{ij}) = M_{i,j}$ is called the $(i,j)$-minor.

This formula for determinant is called the cofactor expansion of a determinant about the $i$th row. There is nothing special about row, we could have also done it using the columns instead:

$$|A| = \text{det}(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \quad \text{det}(A_{ij}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \quad M_{i,j}$$

Problem 1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 2 \end{bmatrix}$. a) Compute $|A|$ by hand.

b) Compute $|A|$ using maple, (enter the command: \text{det}(A);)
c) Compute $|A|$ using the cofactor expansion about the: (i) second row, (ii) the third column.
Problem 2. Let \( A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \).

a. Find \( \det(A) = \).

b. Find \( |A| \) using the cofactor expansion about (i) the first column, (ii) the fourth row.

Problem 3. a) Let \( A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \), find \( \det(A) = \).

b) What do you conclude? The determinant of a diagonal matrix =

c) Let \( B = \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \), find \( \det(B) = \).

d) Let \( C = \begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix} \), find \( \det(C) = \).

e) What do you conclude? The determinant of an upper triangular or a lower triangular matrix =
Problem 4. For the matrices

\( A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \)  \( B = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 5 & 8 \\ 3 & 2 & 7 \end{bmatrix} \)

compute:

a) \( |A| = \) , \( |B| = \)

b) \( |A^4| = \) , \( |B^3| = \)

c) \( |A^{-1}| = \) , \( |B^{-1}| = \)

d) What is your conclusion?
For any matrix \( A \), \( \det(A^4) = \)
For any invertible matrix \( A \), \( \det(A^{-1}) = \)
e) Read Theorem 9, Section 6.2.

Problem 5. Let \( A = \begin{bmatrix} 1 + a & 2 + b & 3 + c \\ -1 & 2 & -1 \\ 3 & 1 & 8 \end{bmatrix} \), \( B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -1 \end{bmatrix} \), \( C = \begin{bmatrix} a & b & c \\ -1 & 2 & -1 \\ 3 & 1 & 8 \end{bmatrix} \).

a) Find: \( \det(A) = \)

\( \det(B) = \)

\( \det(C) = \)

\( \det(B) + \det(C) = \)
b) What can you conclude? Can you prove it?

c) Read the Sum of Rows (Columns) Theorem (Theorem 6) in section 6.2.

Problem 6. Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -1 \\ 3 & 1 & 8 \end{bmatrix} \), \( B = \begin{bmatrix} a & 2a & 3a \\ -1 & 2 & -1 \\ 3 & 1 & 8 \end{bmatrix} \), \( C = \begin{bmatrix} a & 2 & 3 \\ -a & 2 & -1 \\ 3a & 1 & 8 \end{bmatrix} \).

a) Find: \( \det(A) = \)

\( \det(B) = \)

\( \det(C) = \)

b) What can you conclude? Can you prove it?

c) Read Theorem 1.

Problem 7. Let \( A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 8 & 1 \\ 7 & 3 & 2 \end{bmatrix} \), \( B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 8 & 1 \\ 7 & 2 & 3 \end{bmatrix} \), \( C = \begin{bmatrix} 7 & 3 & 2 \\ 2 & 1 & 8 \\ 1 & 1 & 1 \end{bmatrix} \).

a) Find

\( \det(A) = \)

\( \det(B) = \)

\( \det(C) = \)

b) What do you conclude?
c) Read the Sign Reversal Theorem (part 2 of Theorem 1, section 6.1).

Section 6.2: 1, 3, 6, 7, 9, 11, 13, 15, 19, 23, 33, 34, 36, 37, 39.