Math 220 - Linear Transformations - Fall 2000

Definition 1. Let V and W two vector spaces. A function $T : V \to W$ is called a **linear transformation** if it satisfies three properties:

a. $T(0) = 0$,
b. $T(u + v) = T(u) + T(v)$ for any $u, v \in V$
c. $T(cu) = cT(u)$ for any $u \in V$, $c \in \mathbb{R}$.

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(x, y, z) = (x - y + z, x + y - z)$. (a) Show T is a linear transformation. (b) Find all the vectors that map to $(0, 0)$ (that is, $T(x, y, z) = (0, 0)$). (c) Describe the range of T. (d) Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$. What is the relationship between T and A?

Theorem 2. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T(v) = Av$ for any $v \in \mathbb{R}^n$ where*

$$A = [T(e_1), T(e_2), \ldots, T(e_n)]$$

More over

a. $Kernel(T) = \{v \in \mathbb{R}^n : T(v) = 0\} = Null(A)$,
b. $Range(T) = Col\,(A)$

Rotations: $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$R_\theta(x, y) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

rotates each vector counterclockwise θ rad about the origin.

Note: For a more detailed explanation of what matrix transformations on the plane do, please read Section 5.1 carefully.

2. Find the matrix associated with the transformation: $T(x, y, z) = (x + 2z, 3x - y, x + z, y - x)$. Find its Kernel and Range.
3. Show that $T(x, y) = (2x - 3y, x + 4y)$ is linear. Find its kernel and range.
4. Is $T(x, y) = (2x - 3y - 3, x + 4y)$ linear?
5. Show that $T : P_2 \to P_1$ is linear, $T(a + bx + cx^2) = b + 2cx$ (that is $T(p) = p'$). Find its kernel and range.
6. Let $A \in M_{2,2}$. Show that $T : M_{2,2} \to M_{2,2}$, $T(B) = AB$ is linear.
7. Let $T : M_{2,2} \to \mathbb{R}^2$, $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b - a \\ c + d \end{bmatrix}$ is linear. Find its kernel and range.
8. Show $T : C[0, 1] \to C[0, 1]$, $T(f)(x) = \int_0^x f(t)dt$ is linear.
9. Show : $\mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not linear.
10. Let $T : P_1 \to P_1$ be a linear transformation such that $T(1 + x) = -7 + 2x$ and $T(-1 + x) = 4 + x$. Find $T(a + bx)$.

$\boxed{\text{NOTE: Exam III is moved to Fri Nov. 17}}$

Homework: Read Sections 5.1 and 5.2, do problems:
Sec 5.1: 1–4, 9–15 odds, 19–29 odds, 34
Sec 5.2: 1, 4, 6, 9, 11, 13, 15, 18, 19, 20, 21, 23, 24, 25, 41.
Sec 5.3: 1, 3, 5, 6, 9, 11, 13, 15