MATH 220 - Change of Basis - Fall 2000

Suppose you have two bases for the same vector space:

\[ \beta = \{v_1, v_2, \ldots, v_n\}, \quad \beta' = \{v'_1, v'_2, \ldots, v'_n\} \]

Each vector in the first basis has a different name in the second basis. Let \( P_{\beta,\beta'} \) be the matrix whose columns are these names

\[ P_{\beta,\beta'} = [[v_1]_{\beta'}, [v_2]_{\beta'}, \ldots, [v_n]_{\beta'}] \]

The is called the change of basis (from basis \( \beta \) to basis \( \beta' \)) matrix.

If \( v \) is any other vector, for which you know the name in the basis \( \beta \), then the name of \( v \) in the basis \( \beta' \) is obtained by doing:

\[ [v]_{\beta'} = P_{\beta,\beta'} [v]_{\beta} \]

1. Let \( \beta \) the standard basis of \( \mathbb{R}^2 \). Let \( \beta' = \{(1, 1), (1, -1)\} \).
   1. Compute the change of basis matrix from \( \beta \) to \( \beta' \).
   2. Compute the change of basis matrix from \( \beta' \) to \( \beta \).
   3. Let \( v = (4, -2) \). Find \([v]_{\beta}\).
   4. Verify \([v]_{\beta'} = P_{\beta,\beta'}[v]_{\beta} \).

2. Let \( \beta \) the standard basis of \( P_3 \). Let \( \beta' = \{1, x, \frac{1}{2} + \frac{3}{2} x^2, -\frac{3}{2} x + \frac{5}{2} x^3\} \).
   1. Compute the change of basis matrix from \( \beta \) to \( \beta' \).
   2. Compute the change of basis matrix from \( \beta' \) to \( \beta \).
   3. Let \( p = x - x^2 + x^3 \). Find \([p]_{\beta'}\).
   4. Let \( p \) be the polynomial \([p]_{\beta'} = (3, 2, 4, 2) \). Find \([p]_{\beta}\).

Note 1: change of basis matrix from \( \beta' \) to \( \beta \) is the inverse of change of basis matrix from \( \beta \) to \( \beta' \).

\[ P_{\beta,\beta'}^{-1} = P_{\beta',\beta} \]

Note 2. If \( \beta' \) is standard basis, then

\[ P_{\beta,\beta'} = [v_1, v_2, \ldots, v_n] \]

Note 3. If \( \beta \) and \( \beta' \) are any given bases, then, \( P_{\beta,\beta'} \) is the last \( n \)-columns of

\[ \text{ref}\left(\begin{bmatrix} [v'_1, v'_2, \ldots, v'_n] & [v_1, v_2, \ldots, v_n] \end{bmatrix}_{\beta'}^{\beta}\right) \]

Homework: Read section 4.5. Do problems 1–19 odds.