RANK ONE TRANSFORMATIONS WITH SINGULAR SPECTRAL TYPE

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Abstract
We show that a certain class of measures arising from generalized Riesz products are singular. In particular, cutting and stacking (i.e. rank one) transformations whose cuts do not grow too rapidly, have singular maximal spectral type. The precise condition is \( \sum_{n=1}^{\infty} (1/w_n^2) = \infty \), where the \( w_n \) is the number of cuts at stage \( n \).

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§1. Introduction.

Let $T$ be a rank one transformation on an interval. Such a transformation may be obtained inductively by the cutting and stacking method. For a detailed exposition of such constructions see Friedman [6]. This class of transformations has proven to be a rich source of examples in ergodic theory for transformations exhibiting different kinds of properties.

Rank one transformations are defined inductively on towers. A tower $H$ is a collection of disjoint intervals of the same length, $\{I_i\}_{i=1}^h$, where $T$ is defined on all but the last interval by just mapping linearly the interval $I_i$ onto the next $I_{i+1}$, and $h$ is called the height of $H$.

\[
\begin{array}{c}
\text{\overbrace{}}^h & I_h \\
\overbrace{T} & I_3 \\
\overbrace{T} & I_2 \\
\overbrace{T} & I_1
\end{array}
\]

Figure 1: Rank one construction.

Start with the initial tower $H_0 = [0, 1]$ of height 1. Let $H_n$ denote the $n^{th}$ tower and $h_n$ its height. Suppose $H_n$ has already been defined. $H_n$ consists of intervals of the same length which form a partition of $[0, r_n]$, for some $r_n > 1$, stacked one on top of the other in some order. To construct $H_{n+1}$, divide the tower $H_n$ into $w_n$ subcolumns of equal width. Then, on top of the $k^{th}$ subcolumn, add a number $a_n(k)$ of consecutive disjoint intervals. The added intervals have the same width as the subcolumns of $H_n$ and are taken to the right of $r_n$. That is, they form a partition of $[r_n, r_{n+1}]$, where $r_{n+1} = r_n + \sum_{k=1}^{w_n} a_n(k) l$, $l$ =length of subcolumns of $H_n$. Then $H_{n+1}$ is the column obtained by stacking these subcolumns one on top of the previous one, starting from the left.

\[
\begin{array}{c}
\begin{array}{c}
\hline
a_n(1) & a_n(2) & \ldots & a_n(k) & a_n(w_n) \\
\hline
B_{n+1} & B_n & \ldots & & \hline
\end{array}
\\
\overbrace{w_n} & \overbrace{h_n}
\end{array}
\]

Figure 2: $(n+1)^{th}$-tower.
By construction, $H_{n+1}$ has height

$$h_{n+1} = w_n h_n + a_n(1) + \cdots + a_n(w_n).$$ (1.1)

We require that $w_n \geq 2$ for all $n$. Also, we require that the total measure be finite, i.e. $\sum_{n=1}^{\infty} |H_n \setminus H_{n-1}| < \infty$, so that $T$ is defined on a finite measure space.

By rescaling by an appropriate constant $r$, we may assume $\cup_{n=0}^{\infty} H_n = [0,1] := X$. Let $B_n$ denote the base of the tower $H_n$. Then

$$B_0 = [0,1/r], \quad B_n = [0, \frac{1}{r w_0 \cdots w_{n-1}}].$$

Thus, we consider $T$ to be defined on $X = [0,1]$, endowed with the Borel sigma algebra and Lebesgue measure. By construction, $T$ is a measure preserving invertible point transformation.

Put $f_n(x) = \frac{1}{\sqrt{|B_n|}}1_{B_n}(x)$ the characteristic function of the $n^{\text{th}}$-base, normalized so that the 2-norm equals 1. Denote by $U_T f$ the operator $U_T f(x) = f(T^{-1} x)$. By construction of $T$, $U_T$ is a unitary operator in $L^2(X)$.

Notice that

$$C = \{ \{T^k(B_n)\}_{k=0}^{h_n-1}\}_{n=0}^{\infty}$$ (1.2)
generates a dense subalgebra of the Borel $\sigma$-algebra, (here we are using the metric (modulo sets of measure zero) given by $d(A,B)$ =Lebesgue measure of $A \triangle B$). Then the subspace generated by the span of $\{U_T^k(f_n) : 1 \leq n < \infty, 0 \leq k < h_n\} = \text{span of } \{1_{T^k(B_n)} : 1 \leq n < \infty, 0 \leq k < h_n\}$ is dense in $L^2(X)$.

§1.1. Spectral measures.

Given $T : X \mapsto X$ a measure preserving invertible transformation, to any $f \in L^2(X)$ there corresponds a positive measure $\sigma_f$ on $S^1$, the unit circle, defined by $\hat{\sigma}_f(n) = \langle U_T^* f, f \rangle$. With the above notation, let $\sigma_n = \sigma_{f_n}$.

**Definition 1.1.** The maximal spectral type of $T$ is the equivalence class of Borel measures $\sigma$ on $S^1$ (under the equivalence relation $\mu_1 \equiv \mu_2$ if and only if $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$), such that $\sigma_f \ll \sigma$ for all $f \in L^2(X)$ and if $\nu$ is another measure for which $\sigma_f \ll \nu$ for all $f \in L^2(X)$ then $\sigma \ll \nu$.

By the canonical decomposition of $L^2(X)$ into decreasing cycles (see appendix in Parry [10]) with respect to the operator $U_T$, there exists a Borel measure $\sigma = \sigma_f$ for some $f \in L^2(X)$, such that $\sigma$ is in the equivalence class defining the maximal spectral type of $T$. By abuse of notation, we will call this measure the maximal spectral type measure, but it can be replaced by any other measure in its equivalence class.

**Lemma 1.2.** $\sigma$ is absolutely continuous with respect to $\sum_{n=0}^{\infty} 2^{-n} \sigma_n$.

**Proof:** Given $f \in L^2(X)$, since the family $C$ defined in (1.2) generates a dense subalgebra, $f$ can be approximated by functions $g_n \in L^2(X)$ which are constant on the levels of the tower $H_n$. Then, $g_n = F_n(U_T) f_n$ for some polynomial $F_n(z)$, and

$$d\sigma_f = d\sigma_{g_n} + d\nu_n = |F_n|^2 d\sigma_n + d\nu_n.$$
where \( \|\nu_n\| \to 0 \) as \( n \to \infty \). Thus, if \( A \) is a set such that \( \sigma_n(A) = 0 \) for all \( n \), then
\[
0 \leq \sigma_f(A) = \nu_n(A) \leq \|\nu_n\| \to 0 \text{ as } n \to \infty.
\]

We will exploit a recurrence relationship between the spectral measures \( \sigma_n \).

The bases \( B_n \)'s are recursively related in the following fashion (see Figure 2):
\[
B_n = B_{n+1} \cup T^{h_n + s_n(1)} B_{n+1} \cup T^{2h_n + s_n(2)} B_{n+1} \cup \ldots \cup T^{(w_n-1)h_n + s_n(w_n-1)} B_{n+1},
\]
where \( s_n(k) = a_n(1) + \cdots + a_n(k) \). Letting
\[
P_n(z) = \frac{1}{\sqrt{w_n}} \sum_{k=0}^{w_n-1} z^{h_n + s_n(k)}
\]  
(1.3)

where \( s_n(0) = 0 \), we obtain \( f_n = P_n(U_T)f_{n+1} \). Iterating this relationship, we have
\[
d\sigma_n = |P_n|^2 d\sigma_{n+1} = \cdots = \prod_{j=0}^{m-1} |P_{n+j}|^2 d\sigma_{n+m}.
\]  
(1.4)
Thus, \( \sigma_n \) is absolutely continuous with respect to \( \sigma_{n+m} \) for all \( m \geq 0 \), and the continuous parts of these two measures are equivalent.

**Theorem 1.3.** Let \( d\rho_n = \prod_{j=0}^n |P_j|^2 d\lambda \). Then \( \tilde{\rho}_n(k) - \tilde{\sigma}_0(k) \to 0 \) as \( n \to \infty \). In other words, (since \( \|\rho_n\| = 1 \)) \( \tilde{\sigma}_0 \) is the weak* limit of the \( \rho_n \). (\( \lambda \) denotes the normalized Lebesgue measure).

**Proof:** A proof of this theorem can be found in Choksi & Nadkarni [5]. Another proof which was kindly supplied by a referee is the following:

Let \( R_n = P_0 \cdots P_n \) and \( Q_n = |R_n|^2 \). To show that \( d\sigma_0 = w^* \lim_{n \to \infty} Q_n d\lambda \) it suffices to show that \( \tilde{\sigma}_0(m) - \hat{Q}_n(m) \to 0 \) as \( n \to \infty \) for all \( m \). But \( d\sigma_0 = Q_n d\sigma_{n+1} \), and
\[
|\tilde{\sigma}_0(m) - \hat{Q}_n(m)| = |\sum_{j=0}^n \hat{Q}_n(m-j)\hat{\sigma}_{n+1}(j)| \leq \sum_{|j| \geq h_{n+1}} |\hat{Q}_n(m-j)|
\]  
(1.5)
because \( |\hat{\sigma}_{n+1}(j)| \leq 1 \) for all \( j \) and \( |\hat{\sigma}_{n+1}(j)| = 0 \) for \( 0 < |j| < h_{n+1} \) by definition of \( \sigma_{n+1} \). On the other hand, by (1.1) and (1.3), \( \text{deg}(P_n) = h_{n+1} - h_n - a_n(w_n) \) (see definition of \( \text{deg} \) in (2.2)). Then by construction,
\[
\text{deg}(R_n) = \text{deg}(P_0) + \cdots + \text{deg}(P_n) \leq \sum_{k=0}^n h_{k+1} - h_k = h_{n+1} - h_0 < h_{n+1}.
\]
Also, \( \text{deg}(Q_n) = \text{deg}(R_n) \), and hence \( \hat{Q}_n(k) = 0 \) for \( |k| \geq h_{n+1} \). Moreover, \( \hat{R}_n(k) = 0 \) or \( (w_0 \cdots w_n)^{-1/2} \) for all \( k \) by definition of \( R_n \). Since \( Q_n = R_n \hat{R}_n \), we have that for \( |k| < h_{n+1} \),
\[
0 \leq \hat{Q}_n(k) \leq \frac{h_{n+1} - |k|}{w_0 \cdots w_n}.
\]  
(1.6)
From (1.5) and (1.6) it follows that if $|m| < h_{n+1},$

$$|\hat{\sigma}_0(m) - \hat{Q}_n(m)| \leq 2 \frac{m^2}{w_0 \ldots w_n} \to 0 \quad \text{as} \quad n \to \infty,$$

finishing the proof. \hfill \blacksquare

The same argument shows that each of the measures $\sigma_n$ enjoys this property, that is, they are the weak* limit of the measures obtained by replacing $d\sigma_{n+m}$ in equation (1.4) by the Lebesgue measure.

We will show that $\sigma_0$ is singular to Lebesgue measure for a class of rank one transformations whose cutting numbers $\{w_n\}_{n=0}^{\infty}$ do not grow too rapidly.

**Theorem 1.4.** If $\sum_{n=1}^{\infty} (1/w_n)^2 = \infty$, then $\sigma_0 \perp \lambda$.

The proof of Theorem 1.4 also shows that $\sigma_n$ is singular to Lebesgue measure for all $n \geq 0$. Then, by Lemma 1.2, $\sigma$ is also singular to Lebesgue measure.

**Corollary 1.5.** If $\sum_{n=1}^{\infty} (1/w_n)^2 = \infty$, then $\sigma \perp \lambda$.

Properties implying the singularity of Riesz products have been studied for some time (see [7], [9], [11], and [12] for some references). In particular it is known that a classical Riesz product is singular if its coefficients are not in $\ell^2$. In §2 we adapt the proof of this result given by Peyrière [11] to the generalized Riesz products $\rho_n$, and thus prove Theorem 1.4. Another proof can be obtained by adapting Bourgain’s approach in [4]. In fact we shall borrow some ideas from both methods.

§1.2. Comments and remarks.

Certain classes of measure preserving transformations are already known to have singular spectral type. For example, Baxter [3] proved that any $\alpha$-rigid transformation, with $\alpha > 1/2$, has singular spectral type. (An $\alpha$-rigid transformation is a measure preserving transformation such that there exists a sequence $\{n_k\}_{k=1}^{\infty}$ for which $\lim_{k \to \infty} m(T^{n_k} A \cap A) \geq \alpha m(A)$ for all measurable sets $A$.)

Naturally, one asks the following question.

**Question 1.6.** Does any $\alpha$-rigid transformation have singular spectral type?

It follows from Theorem 1.4, that for any $0 < \alpha < 1$, one can construct $\alpha$-rigid (but not $\beta$-rigid for $\beta > \alpha$) rank one transformations with singular spectral type. See Example 3.1 below. Moreover, one can construct rank one transformations without rigidity which enjoy this property:

**Proposition 1.7.** There are mixing rank one transformations with singular spectral type.

Bourgain’s result [4] on the spectral type of Ornstein’s mixing rank one transformation already proves this proposition. However, Ornstein’s transformation involves a random construction. In §3, we give a proof of Proposition 1.7 using an explicit construction.

These examples seem to support the suspicion of many that singular spectral type may be a characteristic of rank one transformations.

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Conjecture 1.8. Every rank one transformation has singular spectral type.

A positive answer to this conjecture would be the link between Kalikow’s and Host’s results on the problem of whether 2-fold mixing implies 3-fold mixing, since the first proved it for mixing rank one transformations and the second for transformations with singular spectral type.

Lastly, we mention that the condition in Theorem 1.4 is, of course, not the best possible. Indeed, we can construct transformations violating the hypothesis of the theorem but which have singular spectral type. The simplest example is the rank one transformation obtained by setting \( w_n = n \) and \( a_n(i) = 0 \) for \( 1 \leq i \leq n \), for all \( n \). Since no extra steps are added on any tower, this transformation is rigid (i.e. 1-rigid) and, by Baxter’s condition, has singular spectral type.

A natural conjecture, intermediate between Theorem 1.4 and Conjecture 1.8, is the following:

Conjecture 1.9. If \( \sum_{n=1}^{\infty} c_n = \infty \) for some choices of coefficients \( c_n \) of the polynomials \( |P_n|^2 \), then \( \sigma \perp \lambda \).

§2. Proof of Theorem 1.4.

Lemma 2.1. Let \( \{P_n\}_{n=1}^{\infty} \) be a family of trigonometric polynomials on \( S^1 \) with positive coefficients. Let \( d \rho_n = \prod_{k=1}^{n} |P_k|^2 d \lambda \). If \( \|P_n\|_2 = 1 \) and \( \| \prod_{i=1}^{n} P_i \|_2 = 1 \) for all \( n \), then \( \lim_{n \to \infty} \rho_n \) exists in the weak* topology.

Proof: By the hypothesis, the measures \( \{ \rho_n \}_{n=1}^{\infty} \) satisfy:

(a) \( \rho_n \) is a probability measure on \( S^1 \) for all \( n \),
(b) \( \hat{\rho}_{n+1}(j) \geq \hat{\rho}_n(j) \) for all \( j \) and \( n \).

Indeed, let \( Q_n = \prod_{j=1}^{n} |P_j|^2 \). By hypothesis, \( |P_{n+1}|^2 = 1 + R_n \) where \( R_n \) is a trigonometric polynomial with positive coefficients and \( \hat{R}_n(0) = 0 \). Thus

\[
\hat{\rho}_{n+1}(j) = \hat{Q}_n(j) + \hat{Q}_n \ast \hat{R}_n(j) \geq \hat{Q}_n(j) = \hat{\rho}_n(j).
\]

From (a) and (b) it follows that \( \lim_{n \to \infty} \rho_n \) exists in the weak* topology.

This lemma implies that all of the weak* limits we shall write down later actually exist. So from now on we omit mentioning this fact every time.

Proposition 2.2. Let \( \{P_n\}_{n=0}^{\infty} \) be a sequence of trigonometric polynomials as in Lemma 2.1. The following two conditions are equivalent:

(a) \( \inf \{ \|P_{n_1} \ldots P_{n_k}\|_1; n_1 < n_2 < \ldots < n_k \} = 0 \),
(b) \( \text{the measure } d \mu = w^* \lim_{n \to \infty} \prod_{j=1}^{n} |P_j|^2 d \lambda \) is singular.

Notes:

1. This proposition is the key element in our proof. It allows us to drop to a subsequence, instead of working with the full sequence of polynomials:
Corollary 2.3. Let \( \{P_n\}_{n=1}^\infty \) and \( \mu \) be as in Proposition 2.2. Let \( \{P_n\}_{k=1}^\infty \) be a sub-
sequence and let \( dv = w^* \lim_{k \to \infty} \prod_{j=1}^k |P_n| |P_n|^2 d\lambda \). If \( v \) is singular, so is \( \mu \).

2. The direction (a) \( \Rightarrow \) (b) is also a key element in Bourgain’s proof in [4]. We are using
this idea as well as the implication (b) \( \Rightarrow \) (a).

The proof of this proposition relies heavily on the fact that the measure is the weak*-
limit of such products. One can easily construct a non–singular measure which is a weak*-
limit of functions \( |f_n|^2 \) with \( \|f_n\|^2 = 1 \) and \( \int |f_n(x)| dx \to 0 \) as \( n \to \infty \).

Proof of Proposition 2.2: Let \( \lambda = \text{Lebesgue measure, and } d\mu = w^* \lim_{N \to \infty} \prod_{n=1}^N |P_n|^2 d\lambda \).
By Lemma 2.1, the limit measure \( \mu \) exists and is a probability measure. Also, for any fixed
finite sequence \( n_1 < \ldots < n_k \), the measure \( d\alpha = w^* \lim_{N \to \infty} \prod_{n=1}^N |P_n|^2 d\lambda \) exists and
is a probability measure. Moreover, \( d\mu = f^2 d\alpha \), where \( f = \prod_{n=1}^N P_n \).

(a) \( \Rightarrow \) (b): To prove that \( \mu \perp \lambda \), it suffices to show that for any \( \epsilon > 0 \), there is a set \( E \)
with \( \lambda(E) > \epsilon \) and \( \mu(E^c) < \epsilon \). Let \( 0 < \epsilon < 1 \).
Choose \( n_1 < \ldots < n_k \) such that \( \int f d\lambda < \epsilon^2 \). By Chebyshev’s inequality, the set
\( E = \{f > \epsilon \} \) satisfies:
\[
\lambda(E) \leq \frac{\|f\|}{\epsilon} \leq \frac{\epsilon^2}{\epsilon} = \epsilon,
\]
and
\[
\mu(E^c) = \int_{E^c} d\mu = \int_{E^c} f^2 d\alpha \leq \int_{E^c} \epsilon^2 d\alpha \leq \epsilon^2 < \epsilon.
\]

(b) \( \Rightarrow \) (a): Given \( 0 < \epsilon < 1 \), there exists a continuous function \( \varphi \) such that:
\[
0 \leq \varphi \leq 1, \quad \mu(\{\varphi \neq 0\}) < \epsilon, \quad \text{and } \lambda(\{\varphi \neq 1\}) < \epsilon.
\]
Let \( f_N = \prod_{n=1}^N |P_n| \). Let \( A = \{\varphi \neq 1\} \), then
\[
\int f_N d\lambda = \int_A f_N d\lambda + \int_{A^c} f_N d\lambda
\leq \lambda(A)^{1/2} + \lambda(A^c)^{1/2} \lambda(A^c)^{1/2}
\leq \sqrt{\epsilon} + \left( \int f_N^2 \varphi d\lambda \right)^{1/2}.
\]
But since \( d\mu = w^* \lim_{N \to \infty} |f_N|^2 d\lambda \),
\[
\lim_{N \to \infty} \int f_N^2 \varphi d\lambda = \int \varphi d\mu \leq \mu(\{\varphi \neq 0\}) < \epsilon.
\]
Thus, taking \( N \) sufficiently large, \( \int f_N d\lambda < 2 \sqrt{\epsilon} \). Since \( \epsilon \) is arbitrary, \( \lim_{N \to \infty} \int f_N d\lambda = 0 \).

The next theorem illustrates Peyrière’s technique and is the backbone of the proof of
Theorem 1.4.

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Theorem 2.4. (Peyrière [11]) Let $\mu$ be a Borel probability measure on $S^1$. If there exists an increasing sequence of integers $\{m_k\}_{k=1}^{\infty}$ such that

(a) $\hat{\mu}(m_k) = a_k$, and $\{a_k\}_{k=1}^{\infty} \notin \ell^2$ and

(b) $\hat{\mu}(m_k - m_j) = a_k \delta_{kj}$ if $k \neq j$,

then $\mu \perp \lambda$.

Proof: Let $f_k(z) = z^{m_k}$. Then $\{f_k\}_{k=1}^{\infty}$ is a bounded orthogonal system in $L^2(\lambda)$, and $\{(f_k - \overline{a_k})\}_{k=1}^{\infty}$ is an orthogonal system in $L^2(\mu)$. Also, since $\mu$ is a finite measure, $\|f_k - \overline{a_k}\|_{L^2(\mu)}$ is bounded in $k$.

Let $\{c_k\}_{k=1}^{\infty} \in \ell^2$ be a sequence such that $\overline{a_k} c_k \geq 0$ and $\sum_{k=1}^{\infty} \overline{a_k} c_k = \infty$. Such a sequence exists since $\{a_k\}_{k=1}^{\infty} \notin \ell^2$. Then, the sequences of functions $\sum_{k=1}^{n} c_k f_k$ and $\sum_{k=1}^{n} c_k (f_k - \overline{a_k})$ converge in $L^2(\lambda)$ and $L^2(\mu)$ respectively. Thus, there is a sequence $n_j$ such that

$$\sum_{k=1}^{n_j} c_k f_k \text{ converges } \lambda\text{-a.e. as } j \to \infty,$$

and

$$\sum_{k=1}^{n_j} c_k (f_k - \overline{a_k}) \text{ converges } \mu\text{-a.e. as } j \to \infty.$$

Then, both series cannot converge for the same $z$ because their difference is

$$\sum_{k=1}^{n_j} c_k \overline{a_k} \to \infty.$$

Hence, the set $E$ on which the first series converges is a Borel set such that $\lambda(E^c) = 0$ and $\mu(E) = 0$, which ends the proof.

§2.2. Fourier coefficients of generalized Riesz products.

In light of Theorem 2.4, we need to look at the Fourier coefficients of the generalized Riesz products defining $\sigma_0$.

Recall, from equation (1.3), that the polynomial $P_n$ has the form

$$P_n(z) = \frac{1}{\sqrt{w_n}} (z^{c_0} + z^{c_0+c_1} + z^{c_0+c_1+c_2} + \cdots + z^{c_0+c_1+c_2+\cdots+c_{w_n-1}}),$$

where $c_0 = 0$, $c_i = h_n + a_n(i)$, $i = 1, \ldots, w_n - 1$. That is, $c_1, c_2, \ldots, c_{w_n-1}$ are the heights of the first $w_n - 1$ subcolumns of the tower $H_n$. Now form the product

$$R_n = P_n \overline{P_n} = 1 + \frac{1}{w_n} \sum_{0 \leq i \neq j \leq w_n-1} z^{(c_0+\cdots+c_i)-(c_0+\cdots+c_j)}.$$  \hfill (2.1)

Define the degree of any trigonometric polynomial $f$ by

$$\deg(f) = \max \{|k| : \hat{f}(k) \neq 0\}.$$  \hfill (2.2)

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Let \( d_n = \deg(P_n) = \deg(R_n) \). From equations (1.1) and (1.3) of the introduction, we have
\[
d_n = h_{n+1} - h_n - a_n(w_n) < h_{n+1},
\]
\[
h_n \leq h_{n+1}/w_n \leq h_{n+1}/2.
\]

With the help of Proposition 2.2, instead of working with the full sequence of polynomials \( \{P_n\}_{n=0}^\infty \), we can drop to a subsequence. Indeed, to show that \( \sigma_0 \) is a singular measure, it suffices to show that \( \text{w}^* \lim_{k \to \infty} \prod_{j=1}^k R_{n_j} d\lambda \) is a singular measure for some sequence \( \{n_k\}_{k=1}^\infty \).

Assume \( \{n_k\}_{k=1}^\infty \) is a sequence satisfying \( n_{k+1} \geq n_k + 3 \). Let \( Q_k = R_{n_1} \ldots R_{n_k} \). Then, from (2.3) and (2.4) it follows that
\[
h_{n_k+1} \leq \frac{1}{4} h_{n_k+1},
\]
and telescoping, since \( n_j + 1 \leq n_{j+1} \),
\[
q_k := \deg(Q_k) = d_{n_1} + d_{n_2} + \cdots + d_{n_k}
\leq (h_{n_1+1} - h_{n_1}) + (h_{n_2+1} - h_{n_2}) + \cdots + (h_{n_k+1} - h_{n_k})
\leq h_{n_k+1}.
\]

Now we will look at the Fourier coefficients of the polynomials \( Q_k \). First, we need to examine \( R_{n_k} \).

Under the above hypothesis, the polynomial \( R_{n_k} \) has isolated Fourier coefficients at 0 and \( d_{n_k} \), in the sense that \( \hat{R}_{n_k} = 0 \) on a large interval of integers around 0 and \( d_{n_k} \).

**Lemma 2.5.** If \( \{n_k\}_{k=1}^\infty \) is a sequence such that \( n_{k+1} \geq n_k + 3 \), then the Fourier coefficients of \( R_{n_k} \) satisfy:

(a) \( \hat{R}_{n_k}(0) = 1, \quad \text{and} \quad \hat{R}_{n_k}(n) = 0, \quad 0 < |n| < h_{n_k}, \)

(b) \( \hat{R}_{n_k}(d_{n_k}) = 1/w_{n_k} \) and \( \hat{R}_{n_k}(n) = 0, \quad d_{n_k} - h_{n_k} < |n| < d_{n_k} \).

**Proof:** Recalling that \( R_{n_k} \) is given by (2.1), let \( d_{i,j} = (c_0 + \cdots + c_i) - (c_0 + \cdots + c_j), \) \( i \neq j \). Since \( d_{i,j} \neq 0 \) if \( i \neq j \), it is immediate that \( \hat{R}_{n_k}(0) = 1 \). The rest of (a) follows from the fact that for \( i > j \), \( d_{i,j} \geq c_i \geq h_{n_k} \), and by symmetry, \( |d_{i,j}| \geq h_{n_k} \) for \( i < j \) also.

To prove (b), note that \( d_{n_k} = \max d_{i,j} = d(w_{n_k} - 1, 0) \). This implies our claim that \( \hat{R}_{n_k}(d_{n_k}) = 1/w_{n_k} \).

Lastly, suppose \( i > j \) and \( (i, j) \) is not the pair \( (w_{n_k} - 1, 0) \). Then \( d_{i,j} = c_{j+1} + \cdots + c_i \) is a sum over a proper subset of the indexes \( \{1, \ldots, w_{n_k} - 1\} \). Therefore, \( d_{n_k} - d_{i,j} \), being the sum over the complement of this subset, satisfies \( d_{n_k} - d_{i,j} \geq \min\{c_1, \ldots, c_{w_{n_k} - 1}\} \geq h_{n_k} \).

That is, there is a gap of at least \( h_{n_k} \) between \( d_{n_k} \) and the previous non-zero Fourier coefficient, which proves (b).

**Lemma 2.6.** With the above notation, if \( \{n_k\}_{k=1}^\infty \) is a sequence such that \( n_{k+1} \geq n_k + 3 \), then the Fourier coefficients of the \( \{Q_k\}_{k=1}^\infty \) satisfy:

(a) \( \hat{Q}_{k+m}(n) = \hat{Q}_k(n) \) whenever \( |n| \leq q_k, \quad m \geq 0, \)

(b) \( \hat{Q}_k(0) = 1 \) and \( \hat{Q}_k(d_{n_k}) = 1/w_{n_k}, \)

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Proof: Property (a) and the fact that $\hat{Q}_k(0) = 1$ are immediate consequences of (2.5), (2.6) and part (a) of Lemma 2.5.

Now consider the coefficients of $Q_{k+1} = Q_k R_{n_{k+1}}$ on the interval $[d_{n_{k+1}} - q_k, d_{n_{k+1}} + q_k]$. Since $q_k < h_{n_{k+1}}/4$ (see (2.5) and (2.6)), it is clear that (using Lemma 2.5 (b) for $R_{n_{k+1}}$):

$$\hat{Q}_{k+1}(n + d_{n_{k+1}}) = \hat{Q}_k(n) \frac{1}{w_{n_{k+1}}} \quad \text{for all } n \in [-q_k, q_k].$$

(2.7)

In particular, $\hat{Q}_{k+1}(d_{n_{k+1}}) = \hat{Q}_k(0)/w_{n_{k+1}} = 1/w_{n_{k+1}}$ which proves (b).

Given a sequence $n_1 < n_2 < \ldots$, define $\alpha$ to be the probability measure $d\alpha = w^* \lim_{k \to \infty} \prod_{j=1}^k |P_{n_j}|^2 d\lambda$.

**Lemma 2.7.** Let $\{n_j\}_{j=1}^\infty$ be a sequence satisfying $n_{j+1} \geq n_j + 3$. Then there is a sequence $\{m_j\}_{j=1}^\infty \subset \mathbb{N}$ such that:

(a) $\tilde{\alpha}(\pm m_j) = 1/w_{n_j}$,

(b) $\tilde{\alpha}(m_j \pm m_k) = \tilde{\alpha}(m_j) \tilde{\alpha}(m_k), \quad j \neq k$.

Proof: By definition of $\alpha$ and with the notation preceding the lemma, we have $\tilde{\alpha}(n) = \lim_{k \to \infty} \hat{Q}_k(n)$ for all $n$.

Let $m_k = d_{n_k}$. Then, from Lemma 2.6, it follows that

$$\tilde{\alpha}(n) = \hat{Q}_k(n) \quad \text{whenever } |n| \leq q_k,$$

and that $\tilde{\alpha}(m_k) = 1/w_{n_k}$ which proves (a), since $\alpha$ is real.

To prove (b), let $j < k$ and apply equation (2.7) for $k$ instead of $k + 1$, with $n = \pm m_j$. Noting that $m_j \in$ support of $\tilde{Q}_j \subset$ support of $\tilde{Q}_{k-1} \subset [-q_{k-1}, q_{k-1}]$, we get from equations (2.7) and (2.8) that

$$\tilde{\alpha}(m_k \pm m_j) = \hat{Q}_k(m_k \pm m_j) = \hat{Q}_{k-1}(m_j \pm m_j) \frac{1}{w_{n_k}}$$

$$= \tilde{\alpha}(m_j) \tilde{\alpha}(m_k) = \tilde{\alpha}(m_j) \tilde{\alpha}(m_k),$$

which proves (b).

**Proof of Theorem 1.4:** We will apply Lemma 2.7 to a sequence $\{n_j\}_{j=1}^\infty$ which in addition satisfies $\sum_{j=1}^\infty 1/w_{n_j}^2 = \infty$.

By hypothesis, we have

$$\infty = \sum_{n=1}^\infty \frac{1}{w_n^2} = \left( \sum_{n=3j} + \sum_{n=3j-1} + \sum_{n=3j-2} \right).$$

So, at least one of the three sums is $\infty$. Thus we can choose $\{n_j\}_{j=1}^\infty$ such that $n_{j+1} = n_j + 3$ and $\sum_{j=1}^\infty 1/w_{n_j}^2 = \infty$. By Lemma 2.7, the measure $d\alpha = w^* \lim_{k \to \infty} \prod_{j=1}^k |P_{n_j}|^2 d\lambda$ satisfies the hypothesis of Theorem 2.4. Thus, $\alpha \perp \lambda$, and by Corollary 2.3, $\sigma_0 \perp \lambda$.

**§3. Examples.**
Example 3.1. Any rank one transformation with $\liminf w_n < \infty$ has singular spectral type.

The proof is immediate from Theorem 1.4. This shows that, for example, Chacon’s transformation with $w_n = 2$ and $a_n(1) = 0$, $a_n(2) = 1$, which is 1/2-rigid, has singular spectral type. Similarly, for arbitrarily small $\alpha > 0$ one can construct $\alpha$-rigid (but not $\beta$-rigid for $\beta > \alpha$) rank one transformations with singular spectral type. That is, for $\alpha = 1/M$, let $w_n = M$, $a_n(k) = k - 1$ for $k = 1, \ldots, M - 1$ and $a_n(M) = 0$.

Next, to prove Proposition 1.7, we recall a construction of Adams and Friedman [2].

**Definition 3.2** A staircase construction is a rank one transformation $T$ such that, at each step, the numbers of intervals added, $a_n(k)$, are defined by

$$a_n(1) = a_n(w_n) = 0, \quad a_n(k) = k - 1 \quad \text{for} \quad 2 \leq k \leq w_n - 1.$$  

Thus, every staircase construction is completely determined by the sequence $\{w_n\}_{n=1}^\infty$. We denote such a transformation by $T = T_{\{w_n\}}$.

Let $\{r_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ be two sequences defined inductively as follows. $m_1 = 1$ and $r_1 \geq 3$. Assume that $m_1, \ldots, m_{k-1}$ and $r_1, \ldots, r_{k-1}$ have already been defined. Let $s_j = r_l$ for $m_l \leq j < m_{l+1}$, $l < k - 1$, and $s_j = r_{k-1}$ for $j \geq m_{k-1}$. The transformation $S = T_{\{s_j\}}$ is “uniform Cesaro” (see [2]). In particular, there exists $N$ such that for $n \geq N$

$$\|\frac{1}{n} \sum_{i=0}^{n-1} S^i \chi_I(x) - \lambda(I)\|_1 \leq \frac{1}{k} \lambda(I)$$

for all positive integers $l$ and all levels $I$ in the $m_{k-1}$ tower. Choose $r_k$ such that

$$r_k \geq kN$$

and choose $m_k$ such that the height $h_{m_k}$ of the $m_k$ tower satisfies

$$h_{m_k} \geq k r_k^2.$$

With the sequences $\{r_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ as above, construct $\{w_n\}_{n=1}^\infty$ by putting

$$w_n = r_k \quad \text{for} \quad m_k \leq n < m_{k+1}.$$

**Theorem 3.3.** (Adams & Friedman [2]) The staircase $T = T_{\{w_n\}}$ with $\{w_n\}_{n=1}^\infty$ as above, is mixing.

**Proof of Proposition 1.7:** To define the example, choose a staircase transformation as in Theorem 3.3, with the additional property that $m_{k+1} - m_k \geq r_k^2$. Then

$$\sum_{n=1}^\infty \frac{1}{w_n^2} = \sum_{k=1}^\infty \sum_{n=m_k}^{m_{k+1} - 1} \frac{1}{r_k^2} \geq \sum_{k=1}^\infty 1 = \infty,$$
and we can apply Theorem 1.4.

Another example which proves Proposition 1.7 is the standard staircase transformation \((w_n = n)\). This transformation has recently been shown to be mixing by Adams [1], and also to have singular spectral type by Klemes [8].

REFERENCES: