ON APPROACH REGIONS FOR THE CONJUGATE POISSON INTEGRAL
AND SINGULAR INTEGRALS.

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Abstract

Let \( \tilde{u} \) denote the conjugate Poisson integral of a function \( f \in L^p(R) \). We give conditions on a region \( \Omega \) so that \( \lim_{(v,\epsilon) \to (0,0)} \tilde{u}(x + v, \epsilon) = Hf(x) \), the Hilbert transform of \( f \) at \( x \), for a.e. \( x \). We also consider more general Calderón–Zygmund singular integrals and give conditions on a set \( \Omega \) so that \( \sup_{(v,\epsilon) \in \Omega} \left| \int_{|t| > \epsilon} k(x + v-t)f(t)dt \right| \) is a bounded operator on \( L^p \), \( 1 < p < \infty \), and is weak \((1,1)\).

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Let \( f \in L^p(R^d) \) and let \( u(x, y) \) denote the Poisson integral of \( f \). Then a classical theorem of Fatou [3] asserts that \( u \) has non-tangential limits a.e. on \( R^d \). In 1984, Nagel and Stein [5] considered more general convergence than the classical non-tangential convergence and gave necessary and sufficient conditions for an approach region \( \Omega \) so that convergence occurs if \( u(x, y) \) approaches the boundary through the region \( \Omega \).

In this paper we consider the associated problem for the conjugate Poisson integral of a function \( f \), as well as for more general Calderón–Zygmund singular integrals.

Let \( k(x) \) be a Calderón–Zygmund kernel on \( R^d \), that is, \( k(x) = \frac{w(x)}{|x|^d} \), where:

1. \( w \) is homogeneous of degree 0 and \( w \in L^\infty(S^{d-1}) \),
2. the integral over the \( S^{d-1} \) sphere vanishes, and
3. \( |k(x + y) - k(x)| \leq C |y|/|x|^{d+1} \), if \( |x| > 2|y| \).

Let \( k_1(x) = k(x) \) if \( |x| > 1 \) and 0 otherwise, and define \( k_r(x) = r^{-d} k_1(x/r) \). Consider the \( d \)-dimensional singular integral defined by this kernel,

\[
H_r f(x) = \int_{|x-t| > r} f(t) k(x-t) dt = f * k_r(x).
\]

Given a set \( \Omega \subset R^d \times R^+ \), consider the maximal transform

\[
H^\#_\Omega f(x) = \sup_{(v, r) \in \Omega} |H_r f(x + v)|.
\]

We will also use the notation

\[
H^\# f(x) = \sup_{r > 0} |H_r f(x)|, \quad \text{ where } H f(x) = \lim_{r \to 0} H_r f(x),
\]

and the standard Hardy–Littlewood maximal function

\[
M f(x) = \sup_{r > 0} \frac{1}{|B(0, r)|} \int_{B(0, r)} |f(x + t)| dt.
\]

In this paper we find necessary and sufficient conditions on the sets \( \Omega \) for which \( H^\#_\Omega f \) is a weak \((1,1)\) and strong \((p,p)\) operator, \( 1 < p < \infty \). It turns out that such sets coincide with those \( \Omega\)'s for which the moved Hardy–Littlewood's maximal operator

\[
M_\Omega f(x) = \sup_{(v, r) \in \Omega} \frac{1}{|B(v, r)|} \int_{B(v, r)} |f(x + t)| dt
\]

is a weak \((1,1)\) and strong \((p,p)\) operator, \( 1 < p < \infty \). Nagel and Stein [5] showed that a necessary and sufficient condition for \( M_\Omega f \) to be weak \((1,1)\) and strong \((p,p)\), \( 1 < p < \infty \), is that the set \( \Omega \) satisfies the following condition, known as the cone condition.

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Definition 1. We say that a set $\Omega \subset R^d \times R^+$ satisfies the cone condition if for any $\alpha$, the set

$$\Omega_\alpha = \{(x, y) \in R^d \times R^+ : \exists (v, r) \in \Omega \text{ such that } |x - v| < \alpha(y - r)\}$$

has the property that there exists a constant $C = C(\alpha)$ such that the cross section set

$$\Omega_\alpha(\lambda) = \{x \in R^d : (x, \lambda) \in \Omega_\alpha\}$$

satisfies

$$|\Omega_\alpha(\lambda)| \leq C\lambda^d,$$

for all $\lambda > 0$.

In the first section we show that if $\Omega$ satisfies the cone condition then $H^\#_\Omega f$ and $\sup_{(v, r) \in \Omega} |Q_r * f(x + v)|$, the maximal function associated with conjugate Poisson kernel, are weak (1,1) and strong (p,p) operators, $1 < p < \infty$. The sufficiency of the cone condition in the one dimensional case was already proved in S. Ferrando's Ph.D. thesis [4]. Ferrando reduced the problem to the case in which $\Omega$ is a discrete set and proved the result using a covering argument plus a discrete version of the Hilbert transform. In this work, we extend the result to $R^d$ by using an argument involving atomic decompositions for functions in $R^d_+$. In section 2, we show that the cone condition is also necessary for $H^\#_\Omega$ to be weak (p,p), $1 \leq p < \infty$, when $k(x)$ is any of the Riesz kernels. In section 3, we show the existence of the limit of $H_r f(x + v)$ as $(v, r)$ approach $(0,0)$ on a region satisfying the cone condition. We apply this result to the convergence of $Q_y f(x + v)$, the conjugate Poisson integral of $f$, when $(v, y)$ tends to $(0,0)$ on an approach region $\Omega$ satisfying the cone condition. Lastly, in section 4, we apply the results to the Ergodic theory setting.

§1. Maximal estimates.

The proof that the maximal operator $H^\#_\Omega$ is weak (1,1) and strong (p,p) for $1 < p < \infty$, will make use of the atomic decomposition for operators in $R^d_+$. This approach was suggested to us by E. M. Stein, greatly simplifying our original proof.

The atomic decomposition allows us to reduce the problem of showing that $H^\#_\Omega f$ is a weak (1,1) and strong (p,p), $1 < p < \infty$, to showing that a simpler operator is of the same type.

Let $\bar{\Omega} = \{(x, y) : (x, y_0) \in \Omega \text{ for some } y_0 \leq y\}$. Then $H^\#_\bar{\Omega} f(x) \geq H^\#_\Omega f(x)$, and if $\Omega$ satisfies the cone condition, so does $\bar{\Omega}$ because $\bar{\Omega}_\alpha = \Omega_\alpha$ (see Definition 1). Therefore, there is no harm in working with the extended set $\bar{\Omega}$ instead, which simplifies the proof.

Let $\Gamma = \{(v, t) \in R^d_+ : |v| < t\}$. That is, $\Gamma$ is a single cone positioned at $(0,0)$. Then $H^\#_\Gamma f(x) = \sup_{(v, r) \in \Gamma} |H_r f(x + v)|$ is the standard nontangential maximal function for the associated singular integral operator.

Theorem 2. If $\Omega$ satisfies the cone condition, then

a) $\int_{R^d} |H^\#_\Omega f(x)|^p dx < c_p \int_{R^d} |H^\#_\Omega f(x)|^p dx, \text{ for } 0 < p < \infty$, 

b) $\{|x \in R^d |H^\#_\Omega f(x) > \lambda\| \leq c|\{x \in R^d |H^\#_\Omega f(x) > \lambda\|$, 

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c) $H_\Omega^\# f(x) \leq H^\# f(x) + C(d)Mf(x)$, and

d) $H_\Omega^\# f$ is a weak $(1,1)$ and strong $(p,p)$ operator, for $1 < p < \infty$.

Proof: Parts a) and b) are an application of the results contained in Stein’s Harmonic Analysis [7], pages 68 and 69. For completeness, we include his argument.

a) An atom associated to a ball $B \subset R^d$, is a measurable function $a(x,t)$ supported in the tent $T(B) = \{(x,t) : |x| < r - t \} \subset R^d_+$, such that $\|a\|_\infty \leq 1/|B|$.

If $H_\Omega^\# f(x) \in L^p(R^d)$ then we can apply the atomic decomposition to the function $|H_y f(x)|^p$. Hence, to prove a), it will be enough to consider the case where $p = 1$ and $H_y f(x) = a(x,y)$ is an atom. Further, by translation, we can assume that atom is supported in $T(B)$, where $B$ is a ball of radius $r$ centered at the origin.

By the properties of the atom $a$, we clearly have $\sup_{(v,y) \in \Omega} |a(x+v,y)| \leq 1/|B|$. If $\sup_{(v,y) \in \Omega} |a(x+v,y)| \neq 0$ then there is a $(v,y) \in \Omega$ such that $(x+v,y) \in T(B)$; that is, $|x+v| < r-y$. Since $(v,y) \in \Omega$, then $-x \in \Omega_1(r) = \Omega_1(r)$. Hence

$$[\{x \mid \sup_{(v,y) \in \Omega} |a(x+v,y)| \neq 0\}] \leq |\Omega_1(r)|,$$

and by assumption, $|\Omega_1(r)| \leq cr^d$. From this we get

$$\int_{R^d} \sup_{(v,y) \in \Omega} |a(x+v,y)| dx \leq \frac{1}{|B|} |\Omega_1(r)| \leq c. \quad (1)$$

Since (1) holds for atoms, a) holds in general (by Theorem 3.2.3 in [7]).

b) To prove b) we repeat the same proof, but replace the function $H_y f(x)$ by the characteristic function of the set where $|H_y f(x)| > \lambda$.

c) It is easy to see that the operator $H_\Omega^\# f$ can be compared the maximal operator $H^\# f$. Indeed

$$|H_r f(x+v) - H_r f(x)| \leq \int_{|t| > 2r} |f(x-t)||k(t-v) - k(t)| dt$$

$$+ \int_{|t| > 2r} |f(x-t)||k(t)| dt + \int_{r < |t| \leq 2r} |f(x-t)||k(t)| dt.$$

By property (k1), $|k(x)| \leq c/|x|^d$, thus the last two terms are majorized by

$$c(d) \frac{1}{|B(0,2r)|} \int_{B(0,2r)} |f(x-t)| dt.$$

To handle the first term, recall that by (k3), $|k(t-v) - k(t)| \leq C|v|/|t|^{d+1}$, if $|t| > 2|v|$. Thus, if $|v| < r$,

$$|k(t-v) - k(t)| \leq C \frac{r}{|t|^{d+1}} = C\Phi_r(t), \quad \text{for } |t| > 2r,$$

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where $\Phi_r(t) = r^{-d} \Phi_1(t/r)$, and $\Phi_1(t) = |t|^{-d-1}$ for $|t| > 2$. Thus

$$\sup_{(v, r) \in \Gamma} |H_r f(x + v) - H_r f(x)| \leq C \sup_{r > 0} |f| \Phi_r(x) + c(d) M f(x).$$

Since $\Phi_1$ is an integrable function on $R^d$ which radially decreases at infinity with an appropriate rate, $\sup_{r > 0} |f| \Phi_r(x)$ is also dominated by $M f(x)$. Hence

$$\sup_{(v, r) \in \Gamma} |H_r f(x + v)| \leq \sup_{r > 0} |H_r f(x)| + C(d) M f(x),$$

finishing the proof of c).

d) The proof of d) is an straight forward application of a), b) and c). □

Let $Q_y(x) = \frac{1}{\pi} \frac{x}{r^2 + y^2}$ denote the conjugate Poisson kernel in $R^2_+$. For a set $\Omega \subset R^2_+$, let $Q^\#_y(x) = \sup_{(v, t) \in \Gamma} |Q_t \ast f(x + v)|$. With this notation, the corresponding version of Theorem 2 also holds for this maximal operator.

**Theorem 3.** If $\Omega$ satisfies the cone condition, then

a) $\int_{R^d} |Q^\#_y(x)|^p dx < c_p \int_{R^d} |Q^\#_y(x)|^p dx$, for $0 < p < \infty$,  
b) $|\{x \in R^d | Q^\#_y(x) > \lambda\}| \leq c |\{x \in R^d | Q^\#_y(x) > \lambda\}|$,  
c) $Q^\#_y(x) \leq \frac{1}{\pi} |H^\#_y f(x) + c(d) M f(x)|$, and  
d) $H^\#_y f$ is a weak $(1,1)$ and strong $(p,p)$ operator, for $1 < p < \infty$.

**Proof:** The proof is exactly the same as the proof of Theorem 2. □

**§2. Necessity of the cone condition.**

The Riesz kernels in $R^d$ are defined by the $j$th coordinate in the following way:

$$k_j(x) = \frac{w_j(x)}{|x|^d}, \text{ where } w_j(x) = \frac{x_j}{|x_j|}.$$

**Proposition 4.** Let $k$ be a Riesz kernel in $R^d$. If $H^\#_k f$ is weak $(p,p)$ for some $1 \leq p < \infty$ then $\Omega$ satisfies the cone condition.

**Proof:** Recall

$$\Omega_\alpha = \{(x, t) : \exists (v, r) \in \Omega : |x - v| < \alpha(t - r)\}.$$

Without loss of generality we can assume $k(x) = k_1(x)$. For a fixed $\alpha$, we need to estimate the measure of $\Omega_\alpha(\lambda) = \{x : (x, \lambda) \in \Omega_\alpha\}$ for any $\lambda > 0$.

Let $b \geq 2\alpha\lambda$ to be determined and

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x_1 \leq b \text{ and } |x_i| \leq \alpha\lambda \text{ for all } 2 \leq i \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

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Let $x \in \Omega_{\alpha}(\lambda)$ and $(v, r) \in \Omega$ such that $|x - v| < \alpha(\lambda - r)$. Then
\[
|H_{r}f(v - x)| = \left| \int_{|t| > r} f(t - (v - x)) \frac{w_{1}(t)}{|t|^{d}} \, dt \right|
= \int_{|t| > r, \begin{array}{c} |t_{1}| + |v_{1} - x_{1}| < b + |v_{1} - x_{1}|, \vspace{1ex} \\
|t_{i} - (v_{i} - x_{i})| < \alpha \lambda, i \neq 1 \end{array}} \frac{1}{|t|^{d}} \, dt
\]
by the symmetry of the kernel.

Case 1: $r < \alpha \lambda$.

![Figure 1](attachment:image1.png)

In this case, since $|x - v| < \alpha \lambda$,
\[
H_{r}f(v - x) \geq \int_{|t| > r, \begin{array}{c} \alpha \lambda < |t_{1}| < b - \alpha \lambda \\
|t_{i} - (v_{i} - x_{i})| < \alpha \lambda, i \neq 1 \end{array}} \frac{1}{|t|^{d}} \, dt
\geq c(d) \frac{(b - 2\alpha \lambda)(\alpha \lambda)^{d-1}}{(b + d\alpha \lambda)^{d}}
= c(d) \frac{1}{(3 + d)^{d}},
\]
if $b = 3\alpha \lambda$.

Case 2: $r \geq \alpha \lambda$

![Figure 2](attachment:image2.png)

Now we will use also the fact that $|x - v| \leq \alpha(\lambda - r)$, so in particular, $r \leq \lambda$ and $\alpha \leq 1$. 

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\[ H_r f(v - x) \geq \int_{|t_1 < x_1 < b < a \lambda |} \frac{1}{|t|^d} dt \]
\[ \geq c(d) \frac{(b - a \lambda - r)(a \lambda)^{d-1}}{(b + d a \lambda)^d} \]
\[ \geq c(d) \frac{(b - 2 \lambda)(a \lambda)^{d-1}}{(b + d \lambda)^d} \]
\[ = c(d) \alpha^{d-1} \frac{1}{3 + d)^d}, \]

if \( b = 3 \lambda \).

Let \( A(\alpha) = c(d)/(3 + d)^d \) if \( \alpha \geq 1 \), and \( A(\alpha) = c(d)\alpha^{d-1}/(3 + d)^d \) if \( 0 < \alpha < 1 \). Then, if \( H^\# f \) is a weak \((p,p)\) operator,
\[
|\Omega_\alpha(\lambda)| = \left| \left\{ x : \exists (v,r) \in \Omega : |x - v| < \alpha(\lambda - r) \right\} \right|
\leq \left| \left\{ x : \sup_{(v,r) \in \Omega, r \leq \lambda} H_r f(v - x) > A(\alpha) \right\} \right|
\leq \left| \left\{ x : H^\# \Omega f(-x) > A(\alpha) \right\} \right|
\leq \frac{C}{A(\alpha)^p} \|f\|_p^p = C(d, \alpha) \lambda^d.
\]

Hence \( \Omega \) satisfies the cone condition.

§3. **Almost everywhere convergence along \( \Omega \).**

Let \( \Omega \) satisfy the cone condition. In this section we prove pointwise convergence of
\[
\lim_{(v,r) \to (0,0)} H_r f(x + v) \quad \text{for any} \quad f \in L^p(R^d), \quad 1 \leq p < \infty.
\]

**Theorem 5.** Let \( \Omega \subset R^d \times R^+ \) satisfy the cone condition, such that \( (0,0) \in \Omega \). Then, for any \( f \in L^p(R^d), \quad 1 \leq p < \infty \), \( \lim_{(v,r) \to (0,0)} H_r f(x + v) = Hf(x) \) a.e..

**Proof:** Let \( C_c^1(R^d) \) be the set of functions with compact support and continuous partial derivatives. Let \( f \in C_c^1(R^d) \). Then
\[
H_r f(x + v) = f * k_1(x + v) + \int_{\{r < |x-y| < 1\}} f(y + v) k(x - y) dy
\]
\[ = I(x, v, r) + II(x, v, r). \]

By continuity of \( f \) and compactness of its support, \( I(x, v, r) \to f * k_1(x) \) as \( (v, r) \to (0,0) \).

For the second term, notice that by \( (k2) \), \( \int_{\{r < |x-y| < 1\}} k(x - y)dy = 0 \), thus
\[
II(x, v, r) = \int [f(y + v) - f(x + v)] k(x - y) \chi_{\{r < |x| < 1\}} (x - y) dy.
\]

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Since the differential of $f$ is continuous of compact support, the integrand is majorized by

$$c\ |x - y|^{-d+1} \chi_{\{0 < |w| < 1\}}(x - y)$$

which is integrable. And, as $(v, r) \to (0, 0)$, the integrand converges to

$$[f(y) - f(x)] \ k(x - y) \ \chi_{\{0 < |w| < 1\}}(x - y).$$

From these two estimations,

$$\lim_{(v, r) \to (0, 0)} H_r f(x + v) = H f(x) \quad \text{for all } x.$$  

Now let $f \in L^p(R^d)$. Given $\epsilon > 0$ choose $g \in C^1_c(R^d)$ such that $\|f - g\|_p < \epsilon$. Let

$$\Lambda f(x) := \left| \limsup_{(v, r) \to (0, 0)} H_r f(x + v) - \liminf_{(v, r) \to (0, 0)} H_r f(x + v) \right|.$$  

Then, $\Lambda f = \Lambda(f - g)$ and, by Theorem 2,

$$\left| \{x : \Lambda f(x) > \alpha\} \right| = \left| \{x : \Lambda(f - g)(x) > \alpha\} \right| \leq \frac{C(d)}{\alpha^p} \|f - g\|_p^p \leq \frac{C(d)}{\alpha^p} \epsilon^p.$$  

Since $\epsilon$ is arbitrary, the limit

$$\lim_{(v, r) \to (0, 0)} H_r f(x + v)$$

exists for almost every $x$.

Similar arguments show that $\lim_{(v, r) \to (0, 0)} H_r f(x + v) = H f(x)$ a.e..

**Theorem 6.** Let $Q_\epsilon(x) = \frac{1}{\pi x^2 + y^2}$ denote the conjugate Poisson kernel in $R^2_+$. If $\Omega$ satisfies the cone condition, then

$$\lim_{(v, r) \to (0, 0)} Q_\epsilon * f(x + v) \quad \text{exists for a.e. } x,$$

and is equal to $H f(x)$.

**Proof:** It follows from Theorem 3 and the fact that

$$\lim_{\epsilon \to 0} Q_\epsilon * f(x) = H f(x),$$

(by arguments similar to those in Theorem 5).

§4. Hilbert transform for measurable flows.
Let \((X, \beta, m)\) be a \(\sigma\)-finite measure space and \(\{\tau_t\}_{t \in \mathbb{R}^d}\) a measure preserving action of \(\mathbb{R}^d\) acting on \(X\), which is jointly measurable from \(\mathbb{R}^d \times X\) to \(X\). We now will consider the truncated ergodic singular integrals

\[
H_r f(x) = \int_{r<|t|<1/r} f(\tau_t x) k(t) dt, \quad f \in L^p(X)
\]

and the related moving maximal operator

\[
H^\#_{\Omega} = \sup_{(v, r) \in \Omega} |H_r f(\tau_v x)|.
\]

The singular integral results obtained in section 1 can be translated to this setting by means of a Calderón transfer principle. However, we first need to establish a modified version of the results in section 1, for the truncated singular integrals.

Since we are interested in the limit when \((v, r) \to (0, 0)\), in this section we will assume that for all \((v, r) \in \Omega\), we have \(r \leq 1\).

**Corollary 7.** Let \(\Omega \subset \mathbb{R} \times \mathbb{R}^+\) satisfy the cone condition. Then

\[
\sup_{(v, r) \in \Omega} \left| \int_{r<|t|<1/r} f(x + v + t) k(t) dt \right|
\]

is a weak \((1,1)\) and a strong \((p,p)\) operator for \(1 < p < \infty\).

**Proof:** The result follows from Theorem 2 because

\[
\left| \int_{r<|t|<1/r} f(x + v + t) k(t) dt \right| \leq |H_r f(x + v)| + |H_{1/r} f(x + v)|,
\]

and \(\{(v, 1/r) : (v, r) \in \Omega\}\) satisfies the cone condition if \(r \leq 1\).

**Proposition 8.** *(Transfer principle)* Let \(\Omega \subset \mathbb{R}^d \times \mathbb{R}^+\) and \(1 \leq p < \infty\). If

\[
\sup_{(v, r) \in \Omega} \left| \int_{r<|t|<1/r} \varphi(x + v + t) k(t) dt \right|
\]

is a weak \((p,p)\) operator in \(L^p(\mathbb{R})\), then \(H^\#_{\Omega} f\) is a weak \((p,p)\) operator in \(L^p(X)\).

**Proof:** Fix \(M > 0\) and let \(N = 3M\). Given \(f \in L^p(X)\) define

\[
\varphi_x(t) = \begin{cases} f(\tau_t x) & \text{if } |t| \leq N \\ 0 & \text{otherwise.} \end{cases}
\]

Then, for almost every \(x\), \(\varphi_x \in L^p(\mathbb{R}^d)\). Indeed

\[
\int_X \int_{\mathbb{R}^d} |\varphi_x(t)|^p dt dx = \int_{|t| \leq N} \int_X |f(\tau_t x)|^p dx dt = c(d) N^d \|f\|_p^p.
\]
because the flow is measure preserving.

Let $\Omega_M = \{(v, r) \in \Omega : |v| \leq M, 1/M \leq r \leq M\}$. Then

$$\int_X \{||s| \leq M : \sup_{(v, r) \in \Omega_M} |\int_{r<|s+v-t|<1/r} \varphi_x(t)k(s + v - t)dt| \geq \lambda\} dx$$

$$\leq \frac{C}{\lambda^p} \int_X \|\varphi_x\|_p^p \leq c(d)N^d \frac{C}{\lambda^p} \|f\|_p^p.$$  

Let $A = \{(x, s) \in X \times R^d : \sup_{(v, r) \in \Omega_M} |\int_{r<|s+v-t|<1/r} \varphi_x(t)k(s + v - t)dt| \geq \lambda\}$. Notice that if $(v, r) \in \Omega_M$, $|s| \leq M$ and $|t| < 1/r$, then $f(\tau_{v+s+t}) = \varphi_x(v + s + t)$ because $3M = N$. Thus,

$$\int_X \{s : \sup_{(v, r) \in \Omega_M} |\int_{r<|s+v-t|<1/r} \varphi_x(t)k(s + v - t)dt| \geq \lambda\} dx$$

$$\geq \int_R \int_X \chi_A(x, s) \chi_{\{|x| < M\}}(s) dx ds$$

$$\geq \int |x| < M \sup_{(v, r) \in \Omega_M} |H_r^f(\tau_{v+r})| \geq \lambda) ds$$

$$= c(d)M^d \sup_{(v, r) \in \Omega_M} \{H_r^f(\tau_v) \geq \lambda\}. $$

Since $N = 3M$, we obtain

$$m(x : \sup_{(v, r) \in \Omega_M} |H_r^f(\tau_v)| \geq \lambda) \leq \frac{3^d C}{\lambda^p} \|f\|_p^p.$$  

The proposition follows by letting $M \to \infty$.  

Corollary 9. If $\Omega$ satisfies the cone condition, then $H_r^\#f$ is a weak $(1,1)$ and strong $(p,p)$ operator for $1 < p < \infty$.

Proof: If follows from Corollary 7 and Proposition 8.

Theorem 10. Let $\Omega \subset R^d \times R^+$ satisfy the cone condition and $(0,0) \in \Omega$. Then

$$\lim_{(v, r) \to (0,0) \in \Omega} H_r^f(\tau_v x)$$

exists a.e. for all $f \in L^p(X)$, $1 \leq p < \infty$.

Proof: It suffices to prove that $k_{v, r} \phi(u) := \int_{|t|<1/r} k(t) \phi(u - v - t)dt$ converges in $L^1(R^d)$ as $(v, r) \to (0,0), (v, r) \in \Omega$, for any $\phi \in C^1_c(R^d)$ satisfying $\int_R \phi ds = 0$. Indeed, let

$$O = \{h \in L^1(X) : h(x) = \int g(\tau_r x) \phi(t) dt, g \in L^1(X), \phi \in C^1_c(R^d)\}.$$  

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Then
\[
H'_r h(\tau_{v,x}) = \int g(\tau_{v,x})k_{v,r}\phi(s)\,ds.
\]

The orthogonal complement of \( O \cap L^2(X) \) consists of the invariant functions under the action (see [2]). Thus the theorem would hold for a dense class of functions and then the result follows for all functions by an application of Corollary 9.

Let’s introduce some notation
\[
K_{(v,r)}(s) := \begin{cases} 
\frac{k(s-v)}{r} & \text{if } r \leq |s-v|, \\
0 & \text{otherwise}.
\end{cases}
\]
\[
k_{(v,r)}(s) := \begin{cases} 
\frac{k(s-v)}{r} & \text{if } r \leq |s-v| \leq \frac{1}{r}, \\
0 & \text{otherwise}.
\end{cases}
\]

Hence
\[
\left| \int_{r \leq |s| \leq \frac{1}{r}} f(u - v - s)k(s)\,ds \right| = \left| \int_{r \leq |u - v - s| \leq \frac{1}{r}} f(s)k(u - v - s)\,ds \right|
= \left| \int k_{(v,r)}(u-s)f(s)\,ds \right| = |k_{(v,r)} * f(u)|.
\]

The \( L^1 \)-convergence of \( k_{(v,r)} * \phi \) follows from the following two properties:

(A) \( K_{(v,r)} * \phi \) converges in \( L^1 \), and

(B) \( \|k_{(v,r)} * \phi - K_{(v,r)} * \phi\|_1 \to 0 \) as \( t \to 0 \).

Property (A) follows from Lebesgue’s Dominated Convergence Theorem. We have that \( K_{(v,r)} * \phi \) converges a.e. by Theorem 8. Assume that \( \text{supp}(\phi) \subseteq \{|y| \leq L\} \), then,
\[
|K_{(v,r)} * \phi(u)| \leq \left( c\chi_{\{|y|<2L\}}(u) + \frac{c(d,L)}{|u|^{d+1}}\chi_{R^d \setminus \{|y|<2L\}}(u) \right) \in L^1(R^d).
\]

First consider \(|u| \geq 2L\), then using the basic properties of \( \phi \) and \( K_{(v,r)} \) (recall (k2)), we can compute (for \((v,r)\) small enough),
\[
|K_{(v,r)} * \phi(u)| = \left| \int [K_{(v,r)}(u-s) - K_{(v,r)}(u)]\phi(s)\,ds \right|
\leq \int_{|s| \leq \kappa} |K_{(v,r)}(u-s) - K_{(v,r)}(u)| |\phi(s)|\,ds
\leq \int_{|s| \leq \kappa} |k(u-v-s) - k(u-v)| |\phi(s)|\,ds
\leq c \int_{|s| \leq \kappa} \frac{|s|}{|u-v|^{d+1}} |\phi(s)|\,ds
\leq cc(d) \frac{L^{d+1}}{|u|^{d+1}}.
\]
by (k3). Here $c = c(\phi)$.

Consider now $|u| \leq 2L$, hence taking $(v, r)$ small enough

$$|K_{(v, r)}*\phi(u)| = \left| \int K_{(v, r)}(s)\phi(u - s)ds \right| = \left| \int K_{(v, r)}(s - v)\phi(u + v - s)ds \right|$$

$$= \left| \int_{4L \geq |s| \geq r} \frac{w(s)}{|s|^d}\phi(u - v - s)ds \right|$$

$$c \leq \int_{4L \geq |s| \geq r} \frac{1}{|s|^d}\left|\phi(u - v - s) - \phi(u - v)\right|ds \leq c,$$

where $c = c(\phi)$, because the differential of $\phi$ is continuous of compact support. This ends the proof of (A).

To prove (B), assume supp($\phi$) $\subseteq \{|y| \leq K\}$. By definition of $K_{(v, r)}$ and $k_{(v, r)}$ we have

$$k_{(v, r)}*\phi(u) - K_{(v, r)}*\phi(u) = k_{\frac{1}{r}}*\phi(u - v).$$

Now $K_{\frac{1}{r}}*\phi(u - v) = 0$ if $u \notin S_{(v, r)} := R^d \setminus \{u : |u| < \frac{1}{r} - v - L\}$. We can choose $(u, v)$ small enough such that $u \in S_{(v, r)}$ implies $|u| \geq 2L$, then a similar computation as in (A) gives $|k_{\frac{1}{r}}*\phi(u)| \leq \frac{4}{|u|^{d+1}}$. In summary

$$|k_{(v, r)}*\phi(u) - K_{(v, r)}*\phi(u)| \leq \chi_{S_{(v, r)}}(u)|k_{\frac{1}{r}}*\phi(u - v)| \leq \chi_{S_{(v, r)}}(u)\frac{c}{|u|^{d+1}}.$$

Hence

$$\|k_{(v, r)}*\phi - K_{(v, r)}*\phi\|_1 \to 0.$$  

References


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