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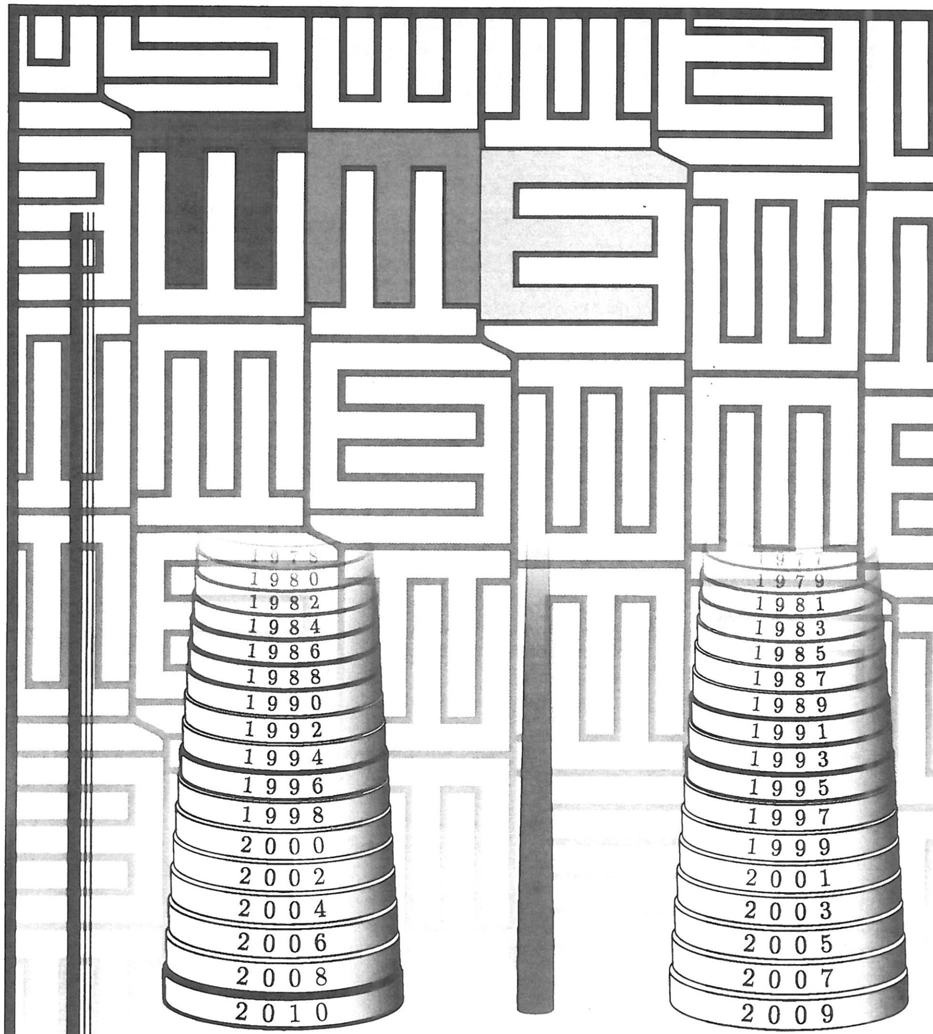

Can you order the discs?
See Problem 1219.

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WAFFLES: IRREDUCIBLE REPRESENTATIONS OF METACYCLIC GROUPS

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Abstract. We present a geometric model for metacyclic groups called a "waffle," and we utilize waffles to find irreducible representations of these groups.

1. Introduction. Representation theory is a bridge between the abstract world of groups and the concrete world of matrix algebra. A representation is a homomorphism that maps every element of a group to some matrix and thus offers a different vantage point from which to study groups.

Every representation is composed of irreducible representations, which are the fundamental "building blocks" of representations much in the same way that prime numbers are the fundamental building blocks of numbers. In the case of $C_3 = \langle a \mid a^3 = 1\rangle$ (and any cyclic group, in fact), the irreducible representations are all of degree 1, meaning each element is represented by a 1×1 complex matrix – i.e. a complex number. Since this group is cyclic, it suffices to define a representation on the generator. Any representation of the group must satisfy the same relations as the group – in this case, $a^3 = 1$ – and thus each of the three irreducible representations of C_3 maps the generator of C_3 to a complex third root of unity. If $\varepsilon = e^{2\pi i/3}$, the three irreducible representations of C_3 are:

$$\rho_1 : a \mapsto 1, \quad \rho_2 : a \mapsto \varepsilon, \quad \rho_3 : a \mapsto \varepsilon^2$$

Because ρ_1 is a homomorphism that maps the generator of C_3 to 1, it maps every element of C_3 to 1. This irreducible representation is called the *trivial representation*, and the trivial representation is one of the irreducible representations of any group, regardless of how many generators the group has. But every nontrivial group also has other irreducible representations, and these are in general more difficult to find, especially when the group has more than one generator or is nonabelian.

In this study we examine the representation theory of metacyclic groups, which have two generators and are nonabelian. The dihedral groups – the groups of rotation and reflection symmetries of regular n-gons – are good examples of metacyclic groups. Understanding the elements of dihedral groups as rotations and reflections of a regular n-gon is a very palpable way to examine dihedral groups. The ability to diagram dihedral groups also turns out to be advantageous in finding their irreducible representations. We extend this insight to a broader class of metacyclic groups and provide a visual "waffle" model for these groups. From the waffle models for metacyclic groups, we are able to obtain some (but not all) of the irreducible representations of these groups. The visualizations of these metacyclic groups are in general not as intuitive as the dihedral visualizations, but in terms of finding irreducible representations, they are just as useful.

2. Metacyclic Groups and Representations. DEFINITION 2.1. *A group G is metacyclic if it contains a cyclic normal subgroup H such that G/H is also cyclic.*

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Metacyclic groups are generally expressed by means of a group presentation of the form

$$G = \langle a, b \mid a^n = b^m = 1, a^s = b^t, bab^{-1} = a^r \rangle$$

where n, m, s, t and r are positive integers subject to the constraints $t \mid m$, $r^t \equiv 1 \pmod{n}$, and $sr \equiv s \pmod{n}$. $H = \langle a \rangle$ is a cyclic normal subgroup, and $G/H \cong C_t$.

NOTE 2.2. *Throughout this study, we will examine metacyclic groups for which $s = n$ and $t = m$, i.e. metacyclic groups with the presentation*

$$G = \langle a, b \mid a^n = b^m = 1, bab^{-1} = a^r \rangle$$

where $r^m \equiv 1 \pmod{n}$ and $|G| = nm$. When we refer to "metacyclic group," we shall mean the subclass of metacyclic groups that have presentations of this form.

We conclude this section with some elementary material about representation theory (cf. [4], [5], [6]). For further background in representation theory, see also [2].

DEFINITION 2.3. *A representation ρ of the group G is a homomorphism $\rho : G \to GL(V)$, where V is some vector space.*

NOTE 2.4. *We will be studying complex representations throughout, and so we take $V \cong \mathbb{C}^d$ for some d, called the degree of the representation.*

The building blocks of representations are the irreducible representations, which are intimately related to invariant subspaces of the vector space V.

DEFINITION 2.5. *Let $\rho : G \to GL(V)$ where V is some vector space, be a representation of G, and let W be a subspace of V. Then W is invariant if $\rho(g) \cdot x \in W$ for all $g \in G$ and $x \in W$, where $\rho(g) \cdot x$ denotes the action of the matrix $\rho(g)$ on the vector $x \in W$.*

With this, we can define what an irreducible representation is.

DEFINITION 2.6. *An irreducible representation $\rho : G \to GL(V)$ is a representation such that V has no proper, nontrivial invariant subspaces.*

A group may have several irreducible representations, just like a number can have many prime factors. It is often desirable to find *every* irreducible representation of a group. The following theorem (see [5] or [6] for a proof) provides a way of counting the number of irreducible representations.

THEOREM 2.7. *The number of irreducible representations of a group G is equal to the number of conjugacy classes of G.*

Thus, finding the conjugacy classes of the group is an important component of finding all the irreducible representations of a group.

3. Making Waffles. A *waffle* is a geometric model for a metacyclic group that encodes information about its conjugacy classes. As such, waffles are useful tools in the search for irreducible representations. The name for these geometric models was inspired by the resemblance of the model for the metacyclic group $SD_{16} = \langle a, b \mid a^{16} = b^2 = 1, bab^{-1} = a^7 \rangle$, shown below, to a waffle:

We begin with a recipe for making waffles:

1. Begin with a metacyclic group $G = \langle a, b \mid a^n = b^m = 1, bab^{-1} = a^r \rangle$.
2. Draw the vertices $e^{2k\pi i/n}$ (for $k = 1, \ldots n$) of a regular n-gon in the complex plane. Vertex $e^{2k\pi i/n}$ will be labeled k and will correspond to the element a^k of G.
3. Based on the conjugation relation $bab^{-1} = a^r$, draw an arrow between conjugate vertices to track the progression of successive conjugations by b. Draw a circle around vertices that represent conjugacy classes with a single element.

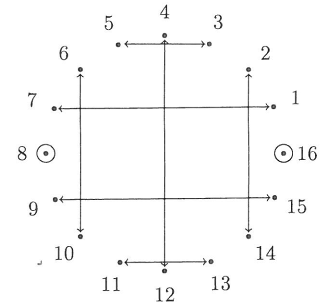


FIG. 1. *The waffle for SD_{16}*

4. The resulting diagram is a waffle for the metacyclic group G.

Since $\langle a \rangle$ is normal, the conjugacy classes of that subgroup are fully contained in the waffle and are an important subset of the conjugacy classes of G.

EXAMPLE 3.1. *The waffle for $D_8 = \langle a, b \mid a^8 = b^2 = 1, bab^{-1} = a^{-1} \rangle$, the dihedral group of order 16, looks like this:*

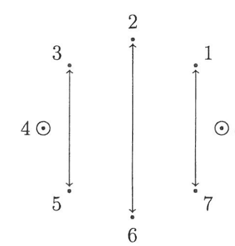


FIG. 2. *The waffle for D_8*

The vertices $k = 1, 2, \ldots, 8$ of the octagon represent the elements a^k of the cyclic subgroup generated by a, and the vertical arrows represent the action of conjugating elements in the waffle by b. These arrows show that $\{a, a^7\}$ forms a conjugacy class, as do $\{a^2, a^6\}$ and $\{a^3, a^5\}$. The element $a^8 = 1$ forms its own conjugacy class, as does a^4, and so these conjugacy classes are indicated by circles.

EXAMPLE 3.2. *The metacyclic group*

$$G_{27} = \langle a, b \mid a^9 = b^3 = 1, bab^{-1} = a^4 \rangle$$

has a somewhat more intricate waffle:

This shows that a, a^4, and a^7 all are in the same conjugacy class, as are a^2, a^8, and a^5. Furthermore, the waffle records the order in which elements of $H = C_9\langle a \rangle$ are conjugated to one another, which will be important later.

EXERCISE 3.3. *Construct the waffle for $SD_8 = \langle a, b \mid a^8 = b^2 = 1, bab^{-1} = a^3 \rangle$. What are the conjugacy classes shown in the waffle? (Hint: there are 5.)*

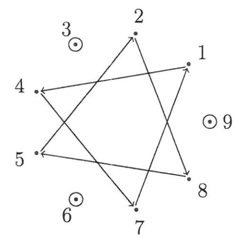


FIG. 3. *The waffle for* G_{27}

EXERCISE 3.4. *Construct the waffles for the Frobenius group of order 20,* $F_{20} = \langle a, b \mid a^5 = b^4 = 1, bab^{-1} = a^2 \rangle$, *and for the group* $F_{21} = \langle a, b \mid a^7 = b^3 = 1, bab^{-1} = a^2 \rangle$.

4. Representations from the Waffle. For a metacyclic group $G = \langle a, b \mid a^n = b^m = 1, bab^{-1} = a^r \rangle$, it suffices to define a representation $\rho : G \to GL(d, \mathbb{C})$ on the generators a and b in such a way that the matrices $\rho(a)$ and $\rho(b)$ satisfy the relations of the group. As we will show, the waffle for a metacyclic group affords an easy way to specify the matrices $\rho(a)$ and $\rho(b)$, and thus waffles give representations of metacyclic groups. Furthermore, the representations they give are irreducible.

Obtaining irreducible representations of a metacyclic group from its waffle requires a slight shift in how we view the waffle. Instead of considering each vertex k of the polygon as a power of the generator of the cyclic normal subgroup, we now consider the waffle in the complex plane and view each vertex as an n^{th} root of unity, where the vertex $k = 1$ corresponds to $e^{2\pi i/n}$ and the vertex $k = n$ corresponds to $1 \in \mathbb{C}$.[1]

NOTE 4.1. *For notational convenience, let* $\varepsilon_n = e^{2\pi i/n}$ *and* $\omega_m = e^{2\pi i/m}$ *throughout, where* n *is the order of* a *and* m *the order of* b.

We now give the method for obtaining an irreducible representation from the waffle:

1. For each vertex k, beginning at $k = 1$ and proceeding counterclockwise around the waffle, let d_k be the number of elements in the conjugacy class of a^k. Then, for each distinct conjugacy class in the waffle, produce the $d_k \times d_k$ matrix $\rho(a)$ as follows:
   (a) If the vertex k represents a conjugacy class with a single element a^k (in which case $d_k = 1$), then map a to the complex number ε_n^k.
   (b) If the vertex k is linked to $d_k > 1$ other vertices by conjugation lines, then map a to the $d_k \times d_k$ diagonal matrix whose entries are the roots of unity corresponding to the d_k elements in the conjugacy class of a^k. The roots of unity are listed down along the diagonal in the same order that b conjugates a^k.
2. Map b to the $d_k \times d_k$ permutation matrix representing the permutation $(1\ d_k\ d_k - 1\ \ldots\ 2)$.

The distinct representations obtained by this method are the irreducible representations of the metacyclic group that are contained in the waffle. Since each conjugacy

[1] We could have the vertex $k = 1$ correspond to any primitive n^{th} root of unity, but we choose $\varepsilon_n = e^{2\pi i/n}$ to make the diagrams easier to use.

class corresponds to an irreducible representation, the number of irreducible representations we obtain from the waffle in this way will be equal to the number of conjugacy classes contained in the waffle.

Because the waffle does not contain all conjugacy classes of the group, we will not be able to read off from the waffle all the irreducible representations of the group. However, in the next section we will discuss how to obtain some of the "missing" irreducible representations. In the last section of the paper we will discuss why the method outlined above produces irreducible representations of metacyclic groups and some of the technical considerations involved in producing all the irreducible representations of metacyclic groups.

Before passing to these considerations, though, we give a few examples of how waffles can be used to produce irreducible representations of metacyclic groups.

EXAMPLE 4.2. *For* D_8 *we can find degree-1 and degree-2 irreducible representations from the waffle. (See the waffle in Example 3.1.)*

The degree-1 irreducible representations in the waffle are indicated by the circled vertices. Picturing the waffle in the complex plane, the circled vertices correspond to 1 and −1. Thus, the degree-1 representations given by the waffle are $a \mapsto 1$ *and* $a \mapsto -1$, *while* b *simply maps to 1. (These degree-1 representations map* b *to 1, since that is the only* 1×1 *permutation matrix.) That is, the degree-1 representations are:*

$$\rho_1 : a \mapsto 1,\ b \mapsto 1; \qquad \rho_2 : a \mapsto -1, , b \mapsto 1$$

The degree-2 irreducible representations in the waffle are given by the conjugation lines connecting two vertices. The element a *maps to a* 2×2 *diagonal matrix with eighth roots of unity as entries, listed down the diagonal in order of conjugation, and* b *maps to the permutation matrix corresponding to the permutation* $(1\ 2)$. *The three degree-2 representations contained in the waffle are:*

$$\rho_3 : a \mapsto \begin{pmatrix} \varepsilon_8 & 0 \\ 0 & \varepsilon_8^7 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rho_4 : a \mapsto \begin{pmatrix} \varepsilon_8^2 & 0 \\ 0 & \varepsilon_8^6 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rho_5 : a \mapsto \begin{pmatrix} \varepsilon_8^3 & 0 \\ 0 & \varepsilon_8^5 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It is straightforward to check that for each value of the index j *the matrices* $\rho_j(a)$ *and* $\rho_j(b)$ *satisfy the relations of the group.*

EXAMPLE 4.3. *The waffle for* G_{27} *contains three degree-1 irreducible representations and two degree-3 irreducible representations. (See the waffle in Example 3.2.)*

The degree-1 irreducible representations in the waffle are given by:

$$\rho_1 : a \mapsto 1, b \mapsto 1 \quad \rho_2 : a \mapsto \varepsilon_9^3, b \mapsto 1 \quad \rho_3 : a \mapsto \varepsilon_9^6, b \mapsto 1$$

The degree-3 irreducible representations in the waffle are given by:

$$\rho_4 : a \mapsto \begin{pmatrix} \varepsilon_9 & 0 & 0 \\ 0 & \varepsilon_9^4 & 0 \\ 0 & 0 & \varepsilon_9^7 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\rho_5 : a \mapsto \begin{pmatrix} \varepsilon_9^2 & 0 & 0 \\ 0 & \varepsilon_9^8 & 0 \\ 0 & 0 & \varepsilon_9^5 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Notice that in the degree-3 irreducible representations, the diagonal entries of $\rho_j(a)$ occur in the same order as the conjugation. Again, it is straightforward to check that for each value of the index j, the matrices $\rho_j(a)$ and $\rho_j(b)$ satisfy the relations of the group.

EXERCISE 4.4. *Find the irreducible representations given by the waffle for SD_8.*

EXERCISE 4.5. *Find the irreducible representations in the waffles for F_{20} and F_{21}.*

5. Finding the Missing Representations using Relations. Since the waffle does not contain every conjugacy class of G, it cannot directly show every irreducible representation. However, it turns out we can use the relations in the presentation of the group, together with some results about metacyclic groups, to find some of the missing irreducible representations. In some cases, all of the irreducible representations may be reconstructed from these results and the relations.

The following theorems are elementary results in representation theory that are often useful in constructing additional irreducible representations when some are known (cf [5], [6]):

THEOREM 5.1. *Let G be a group. If $\rho_1, \dots, \rho_k$ are a complete set of irreducible representations of G of degrees $n_1, \dots, n_k$, respectively, then*

$$\sum_{i=1}^{k} n_i^2 = |G|$$

THEOREM 5.2. *Let G be a group and ρ an irreducible representation of G of degree d. Then d divides $|G|$.*

Using these two results and the relations of the group, we can construct all the irreducible representations of certain metacyclic groups. The next two examples and the two exercises that follow will demonstrate this technique.

EXAMPLE 5.3. *Consider again the dihedral group*

$$D_8 = \langle a, b \mid a^8 = b^2 = 1, bab^{-1} = a^{-1} \rangle$$

The waffle indicates three irreducible representations of degree-2 and two of degree-1, Theorem 5.1 says that the squares of the degrees of the missing representations must sum to 2. Thus, there are two representations missing from the waffle, both of degree-1.

Every degree-1 representation maps group elements to complex numbers, and since $\mathbb{C}$ is abelian, the conjugacy relation will be satisfied trivially. Thus, the only remaining constraint on b in D_8 is the relation $b^2 = 1$. From this we see that in every degree-1 representation, b can map to 1 or -1. Thus, the four degree-1 representations of D_8 are:

$$\begin{array}{lll}
\rho_1 : a \mapsto 1, & b \mapsto 1 & (\textit{in waffle}) \\
\rho_2 : a \mapsto 1, & b \mapsto -1 & (\textit{not in waffle}) \\
\rho_3 : a \mapsto -1, & b \mapsto 1 & (\textit{in waffle}) \\
\rho_4 : a \mapsto -1, & b \mapsto -1 & (\textit{not in waffle})
\end{array}$$

Combined with the three degree-2 irreducible representations of D_8 contained in the waffle (cf. Examples 3.1 and 4.2), this accounts for all the irreducible representations of D_8. Notice that the representations ρ_1 and ρ_2 are lifts from the quotient $D_8/\langle a \rangle$.

Notice also that we produced all irreducible representation of D_8 without computing the conjugacy classes, which can sometimes be painstaking.

EXAMPLE 5.4. *Consider the metacyclic group*

$$G_{27} = \langle a, b \mid a^9 = b^3 = 1, bab^{-1} = a^4 \rangle.$$

The waffle for G_{27} shows three conjugacy classes consisting of a single element and two conjugacy classes that contain three elements. These correspond to three degree-1 and two degree-3 irreducible representations, respectively. (Cf. Examples 3.2 and 4.3.)

Theorem 5.1 indicates that the missing representations must have degrees whose squares sum to 6. Theorem 5.2 tells us further that these degrees must also be divisors of 27. Hence the waffle is missing 6 degree-1 representations. Since $\mathbb{C}$ is abelian, the conjugacy relation $bab^{-1} = a^4$ is again satisfied trivially for degree-1 representations. To find other degree-1 representations, we consider the remaining relation $b^3 = 1$, which has solutions $b = 1, \omega_3, \omega_3^2$. Thus, for each of the degree-1 representations contained in the waffle, b can map to 1, ω_3 or ω_3^2. This gives a total of nine degree-1 representations for G_{27}:

$$\begin{array}{lll}
\rho_1 : a \mapsto 1, & b \mapsto 1 & (\textit{in waffle}) \\
\rho_2 : a \mapsto 1, & b \mapsto \omega_3 & (\textit{not in waffle}) \\
\rho_3 : a \mapsto 1, & b \mapsto \omega_3^2 & (\textit{not in waffle}) \\
\rho_4 : a \mapsto \varepsilon_9^3, & b \mapsto 1 & (\textit{in waffle}) \\
\rho_5 : a \mapsto \varepsilon_9^3, & b \mapsto \omega_3 & (\textit{not in waffle}) \\
\rho_6 : a \mapsto \varepsilon_9^3, & b \mapsto \omega_3^2 & (\textit{not in waffle}) \\
\rho_7 : a \mapsto \varepsilon_9^6, & b \mapsto 1 & (\textit{in waffle}) \\
\rho_8 : a \mapsto \varepsilon_9^6, & b \mapsto \omega_3 & (\textit{not in waffle}) \\
\rho_9 : a \mapsto \varepsilon_9^6, & b \mapsto \omega_3^2 & (\textit{not in waffle})
\end{array}$$

Since G_{27} has 11 conjugacy classes, these nine degree-1 representations and the two degree-3 representations in Example 4.3 comprise all the irreducible representations of G_{27}. Once again, using the relations of the group together with the waffle, we are able to produce all irreducible representations of this metacyclic group. Again, some of these (namely ρ_1, ρ_2 and ρ_3) are lifts from the quotient $G_{27}/\langle a \rangle$.

EXERCISE 5.5. *Find all the irreducible representations of SD_8 (Hint: there are 7.)*

EXERCISE 5.6. *Find all the irreducible representations of F_{20} and F_{21}, the Frobenius groups of order 20 and 21, respectively. (Hint: there are 5 in each case.)*

The astute reader will undoubtedly have noticed that in the previous examples and exercises where we were able to construct all the irreducible representations for the group, the representations missing from the waffle were all of degree-1. The following examples indicate that we cannot hope to be so fortunate all the time.

EXAMPLE 5.7. *Let C_4 denote the cyclic group of order 4 and consider the group*

$$C_4 \ltimes C_4 = \langle a, b \mid a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle.$$

The waffle for $C_4 \ltimes C_4$ produces two degree-1 irreducible representations and one degree-2 irreducible representation. (What are they?) By Theorem 5.1 the missing irreducible representations must have degrees whose squares sum to 10, but without computing the conjugacy classes of $C_4 \ltimes C_4$, it is unclear whether the waffle is missing (i) ten degree-1 representations, or (ii) six degree-1 representations and one degree-2 *representation, or (iii) two degree-1 representations and two degree-2 representations (the only possibilities allowed by Theorems 5.1 and 5.2).*

An even thornier problem is posed by the following example:

EXAMPLE 5.8. *Consider the group*

$$U_{18} = \langle a, b, \mid a^3 = b^6 = 1, bab^{-1} = a^{-1} \rangle.$$

The waffle for U_{18} affords only two irreducible representations, one degree-1 and the other degree-2. (Again, we urge the reader to construct the waffle and produce these representations.) Theorem 5.1 indicates the missing irreducible representations must have degrees that sum to 13, but since the missing representations may be degree-1, -2, or -3, it is unclear which combination we are seeking without actually computing the conjugacy classes of U_{18}.

Some assistance is provided by the following theorems about the number of irreducible representations of degree-1 for metacyclic groups. The first is a general result and may be found in [5] or [4]; the second is specific to metacyclic groups and may be located in [3]. Both of these theorems involve the derived subgroup, which is defined as follows:

DEFINITION 5.9. *Let g and h be elements of a group G. The commutator of g and h is $g^{-1}h^{-1}gh$ and is denoted $[g, h]$. The subgroup of G generated by all the commutators of G is called the derived subgroup of of G and is denoted G'.*

THEOREM 5.10. *Let G be a group. The number of irreducible representations of degree-1 is $|G/G'|$, where G' is the derived subgroup of G.*

THEOREM 5.11. *Let $G = \langle a, b \mid a^n = b^m = 1, a^s = b^t, bab^{-1} = a^r \rangle$ be a metacyclic group. The derived subgroup G' of G is the subgroup $\langle a^{r-1} \rangle$ of order $n/\gcd(n, r-1)$.*

These theorems tell us that in Example 5.7 there are 8 degree-1 irreducible representations for $C_4 \ltimes C_4$, and in Example 5.8 there are 6 degree-1 irreducible representations for U_{18}. By lifting representations of $C_4 \ltimes C_4/\langle a \rangle$ to $C_4 \ltimes C_4$ and using the relations of the group, we are able to produce all 8 degree-1 representations for $C_4 \ltimes C_4$. Similarly, lifting representations of $U_{18}/\langle a \rangle$ to U_{18} and using the relations, we may obtain all 6 degree-1 irreducible representations for U_{18}. Then, by Theorem 5.1, $C_4 \ltimes C_4$ must have one degree-2 irreducible representation in addition to the one given by the waffle, and U_{18} must have two degree-2 irreducible representations in addition to the one given by the waffle.

While waffles are not able to uncover a complete picture of the irreducible representations of metacyclic groups in all cases, they are nonetheless useful tools for producing irreducible representations of metacyclic groups.

6. Leaven for the waffles. In this section we describe an approach to understanding why waffles are such useful tools for determining irreducible representations of metacyclic groups.

Let $G = \langle a, b \mid a^n = b^m = 1, bab^{-1} = a^r \rangle$ be a metacyclic group. Let $H = C_n\langle a \rangle$ denote the cyclic normal subgroup used to create the n-gon for the waffle. Geometrically, we can envision multiplication by a as a rotation of the n-gon counterclockwise by an angle of $2\pi/n$. If the vertices of the n-gon are labeled $1, 2, \ldots n$ counterclockwise, with the n^{th} vertex lying on $1 \in \mathbb{C}$, then the permutation affecting this rotation is $(1\ 2\ \ldots\ n)$. Let A denote the $n \times n$ matrix corresponding to the permutation $(1\ 2\ \ldots\ n)$. The conjugation action of b permutes the vertices of the n-gon according to another permutation dictated by the conjugation relation. Let B denote the $n \times n$ matrix corresponding to this permutation.

Since A corresponds to the permutation $(1\ 2\ \ldots\ n)$, A has n distinct eigenvalues, which are the n^{th} roots of unity, and n distinct eigenvectors v_i, where v_i is the eigenvector corresponding to the eigenvalue ε_n^i. Since A has n distinct eigenvalues, it is diagonalizable. Relative to the eigenbasis $\mathcal{E}$, the diagonal matrix for A is given by

$$[A]_{\mathcal{E}} = \begin{pmatrix} \varepsilon & 0 & \cdots & 0 \\ 0 & \varepsilon^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon^n = 1 \end{pmatrix}.$$

Each eigenvector v_i of A spans a 1-dimensional subspace of $\mathbb{C}^n$ that is invariant under A. The irreducible representations of G correspond to subspaces of $\mathbb{C}^n$ that are invariant under G, i.e. invariant under A and B. Obviously, a subspace of $\mathbb{C}^n$ is invariant under B if and only if it is invariant under B^{-1}. As it will simplify arguments and streamline notation somewhat, we consider B^{-1} acting on $\mathbb{C}^n$ by left multiplication. Left multiplication by B^{-1} defines a right action on the basis vectors v_i of the eigenbasis $\mathcal{E} = \{v_1, \ldots, v_n\}$ for $\mathbb{C}^n$. The following proposition allows us to collect these 1-dimensional subspaces that are invariant under A to form subspaces of dimension d that are invariant under the whole group G. Because these dimension-d subspaces that are invariant under G correspond to irreducible representations of G, we know that d will be a divisor of $|G|$.

PROPOSITION 6.1. $B^{-1}v_i = v_{ir}$.

Proof. Using the defining relations of the group, we have $BAB^{-1} = A^r \iff AB^{-1} = B^{-1}A^r$. Consider the eigenvector v_i. $AB^{-1}v_i = B^{-1}A^r v_i = B^{-1}\varepsilon_n^{ir}v_i = \varepsilon_n^{ir}B^{-1}v_i$. Thus, $B^{-1}v_i$ is an eigenvector of A, and hence $B^{-1}v_i = v_j$ for some j. Since the eigenvalue of $B^{-1}v_i$ is ε_n^{ir}, $B^{-1}v_i = v_{ir}$. □

Proposition 6.1 allows easy access to the subspaces of $\mathbb{C}^n$ that are invariant under G and hence to the irreducible representations of G. The irreducible representations of G correspond to subspaces that are invariant under G and contain no proper subspaces that are also invariant under G. We construct these subspaces inductively. If $\mathbb{C}\langle v_i \rangle$ is invariant under B^{-1}, then $\mathbb{C}\langle v_i \rangle$ corresponds to a 1-dimensional irreducible representation of G. Otherwise, consider $\mathbb{C}\langle v_i, B^{-1}v_i \rangle$, $\mathbb{C}\langle v_i, B^{-1}v_i, B^{-2}v_i \rangle$ and so on until the subspace $\mathbb{C}\langle v_i, B^{-1}v_i, \ldots, B^{-d}v_i \rangle$ is invariant under G.[2] The inductive construction of these subspaces ensures that they will not contain proper subspaces that are also invariant under G. Hence these subspaces correspond to irreducible representations of G.

For each i let $W = \text{span}\,\{v_i, v_{ir}, v_{ir^2}, \ldots, v_{ir^{d-1}}\}$ be a subspace of $\mathbb{C}^n$ that is invariant under G and that corresponds to an irreducible representation of G of dimension d. Further, let $w_j = v_{ir^{j-1}}$ and $\mathcal{W} = \{w_1, w_2, \ldots, w_d\}$ denote the basis for the subspace W. Then, upon restricting B^{-1} to W, we have $B^{-1}|_W(w_j) = w_{j+1 \pmod d}$ and so $B^{-1}|_W$ is represented by the $d \times d$ matrix corresponding to the permutation $(1\ 2\ \ldots\ d)$. The matrix $B|_W$ then corresponds to the permutation $(1\ 2\ \ldots\ d)^{-1} = (1\ d\ d-1\ \ldots\ 2)$, and so

[2]Since $B^m = I$, we know that eventually these subspaces become invariant under G and will have dimension at most m.

$$[B|_W]_{\mathcal{W}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since the $w_j = v_{ir^{j-1}}$ are eigenvectors for the matrix A, the matrix representation for the restriction of $[A]$ to W relative to the basis $\mathcal{W}$ is

$$[A|_W]_{\mathcal{W}} = \begin{pmatrix} \varepsilon^i & 0 & \cdots & 0 \\ 0 & \varepsilon^{ir} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon^{ir^{d-1}} \end{pmatrix}.$$

Hence a maps to the $d \times d$ diagonal matrix with roots of unity listed down the diagonal in the same order that b conjugates a^i.

Thus, the method outlined in Section 4 produces irreducible representations of metacyclic groups.

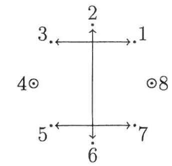
Using the waffle for a metacyclic group $G = \langle a, b \mid a^n = b^m = 1, bab^{-1} = a^r \rangle$ along with the relations of the group and some elementary results from representation theory, many, if not all, of the irreducible representations of the group may be produced.

7. Concluding remarks. The simplicity of waffles makes them an attractive tool for producing irreducible representations of metacyclic groups. However, it also somewhat limits their utility. Ideally waffles would provide all the irreducible representations of any metacyclic group. However, as we currently envision them, waffles do not afford all the irreducible representations (cf. section 5). Furthermore, for a general metacyclic group

$$G = \langle a, b \mid a^n = b^m = 1, a^s = b^t, bab^{-1} = a^r \rangle$$

some of the matrices $\rho(a)$ and $\rho(b)$ predicted by the waffle fail to satisfy the relations of the group, and thus waffles do not always give representations in the general case. This is the case, for instance, with the group of quaternions Q_8, where the matrices predicted by the waffle do not provide representations (cf. [1], [3]). Constructing a geometric model for metacyclic groups that provides all irreducible representations of any metacyclic groups would need to account for these difficulties. Waffles may provide a useful foundation for constructing a more general geometric model that accomplishes these goals. Despite their current limitations, though, waffles are useful in understanding the irreducible representations of metacyclic groups.

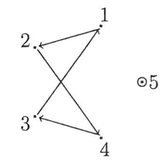
Exercise Solutions. Exercise 3.3:



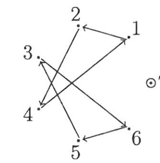
The waffle for SD_8.

Conjugacy classes from the waffle: $\{1\}$, $\{a, a^3\}$, $\{a^2, a^6\}$, $\{a^5, a^7\}$, $\{a^4\}$

Exercise 3.4:



The waffle for F_{20}.



The waffle for F_{21}.

Exercise 4.4: There are five irreducible representations for SD_8 contained in the waffle:

Degree-1:

1. $\rho_1 : a \mapsto 1,\ b \mapsto 1$
2. $\rho_2 : a \mapsto -1,\ b \mapsto 1$

Degree-2: In all degree-2 representations for SD_8, $b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and so we give only the matrices $\rho(a)$.

3. $\rho_3(a) = \begin{pmatrix} \varepsilon_8 & 0 \\ 0 & \varepsilon_8^3 \end{pmatrix}$
4. $\rho_4(a) = \begin{pmatrix} \varepsilon_8^2 & 0 \\ 0 & \varepsilon_8^6 \end{pmatrix}$
5. $\rho_5(a) = \begin{pmatrix} \varepsilon_8^5 & 0 \\ 0 & \varepsilon_8^7 \end{pmatrix}$

Exercise 4.5:

- There are two irreducible representations in the waffle for F_{20}:

  Degree-1:

  1. $\rho_1 : a \mapsto 1, b \mapsto 1$

  Degree-4:

  2. $\rho_2 : a \mapsto \begin{pmatrix} \varepsilon_5 & 0 & 0 & 0 \\ 0 & \varepsilon_5^2 & 0 & 0 \\ 0 & 0 & \varepsilon_5^4 & 0 \\ 0 & 0 & 0 & \varepsilon_5^3 \end{pmatrix}$, $b \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

- There are three irreducible representations in the waffle for F_{21}:

  Degree-1:

  1. $\rho_1 : a \mapsto 1, b \mapsto 1$

  Degree-3: For each of the degree-3 representations of F_{21}, $b \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$; we give the matrices $\rho(a)$:

2. $\rho_2(a) = \begin{pmatrix} \varepsilon_7 & 0 & 0 \\ 0 & \varepsilon_7^2 & 0 \\ 0 & 0 & \varepsilon_7^4 \end{pmatrix}$ 3. $\rho_3(a) = \begin{pmatrix} \varepsilon_7^3 & 0 & 0 \\ 0 & \varepsilon_7^6 & 0 \\ 0 & 0 & \varepsilon_7^5 \end{pmatrix}$

Exercise 5.5: In addition to the five irreducible representations given in Exercise 4.4, there are two additional representations from the relation $b^2 = 1$. From this relation we see that $b = \pm 1$. The representations where $b = -1$ did not appear in the waffle because the matrices representing b had to be permutation matrices. The two additional representations are $\rho_6 : a \mapsto 1,\ b \mapsto -1$ and $\rho_7 : a \mapsto -1,\ b \mapsto -1$.

Exercise 5.6: For F_{20} we are missing three degree-1 representations, according to Theorem 5.1. The relation $b^4 = 1$ gives us that $b = \mapsto \pm 1, \pm i$ for the degree-1 representations; for each of these representations $a \mapsto 1$, as indicated in the waffle. Of these four representations, only the representation $\rho : a \mapsto 1, b \mapsto 1$ is contained in the waffle for F_{20}.

For F_{21} we are missing two degree-1 representations, according to Theorem 5.1. The relation $b^3 = 1$ gives us that $b \mapsto 1, \omega_3, \omega_3^2$ for the degree-1 representations; for each of these representations $a \mapsto 1$, as indicated in the waffle. Of these three representations, only the representation $\rho : a \mapsto 1, b \mapsto 1$ is contained in the waffle for F_{21}.

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PRIMES, TWIN PRIMES AND GOLDBACH'S CONJECTURE

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Numerous problems in Mathematics have remained unsolved for centuries. Two of the most famous are:

Goldbach's Conjecture Every positive even integer $n > 4$ is expressible as the sum of two odd primes.

Twin Prime Problem Are there infinitely many pairs of primes of the form $(p, p+2)$ (hereafter called twin primes)?

The purpose of this note is to show how a given pair of twin primes can be used to generate other twin primes using Goldbach's conjecture.

Let's first look at how Goldbach's conjecture, assuming that it is correct, can be used to show that there are infinitely many primes.

THEOREM 1. *There is an infinite number of primes.*

Proof. Let's assume that there is only a finite number of primes and show that this leads to a contradiction. Let x be an upper bound for the largest prime. By Goldbach's Conjecture there are odd primes p and q such that $p + q = 2x$, but there are only finitely many primes p and q which satisfy this equation. Since there are infinitely many positive even integers $2x$, we have a contradiction. □

Let's now show a method for generating twin primes using Goldbach's Conjecture. Example 1: Starting with the twin prime $(59, 61)$, consider their sum, $59 + 61 = 120$. Now let's use Goldbach's conjecture to find all prime pairs whose sum is 120. They are $(7, 113), (11, 109), (13, 107), (17, 103), (19, 101), (23, 97), (31, 89), (37, 83), (41, 79), (47, 73), (53, 67)$, and $(59, 61)$ From the two components of these twelve ordered pairs of primes, form the following increasing sequence of 24 primes:

$$7, \underline{11, 13}, \underline{17, 19}, 23, 31, 37, 41, 47, 53, \underline{59, 61}, 67, 73, 79, 83, 89, 97, \underline{101, 103}, \underline{107, 109}, 113$$

In this sequence of 24 primes we find five twin primes. They are underlined in the sequence. Hence, using Goldbach's Conjecture and the twin prime pair $(59, 61)$ we were able to generate four other twin primes.

In the following problems, a table of primes (or twin primes) can be helpful to obtain the solutions. The solutions will be used later in Table 1.

Problem 1: Using the twin prime $(11, 13)$, generate all possible twin primes. Solution: $(5, 7)$ and $(17, 19)$.

Problem 2: Using the twin prime $(17, 19)$, generate all possible twin primes. Solution: $(5, 7)$ and $(29, 31)$.

Problem 3: Using the twin prime $(29, 31)$, generate all possible twin primes. Solution: $(17, 19)$ and $(41, 43)$.

Problem 4: Using the twin prime $(41, 43)$, generate all possible twin primes. Solution: $(11, 13)$ and $(71, 73)$.

Problem 5: Using the twin prime $(71, 73)$, generate all possible twin primes. Solution: $(5, 7)$, $(41, 43)$, $(101, 103)$ and $(137, 139)$.

By our method, with each twin prime generator, we have found at least two more twin primes. One of them is the "smallest" another is the "largest". We now create a table by using the "largest" as the generator in the next row. From (11,13) we obtain

Let's stop here and consider two related questions:

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