I. True or false? Please circle your answers.
(1.5 points for each correct answer, but **be careful**: 1 point will be subtracted for each wrong answer!)

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>39</td>
<td>14</td>
<td>11</td>
<td>11</td>
<td>11</td>
<td>14</td>
<td>100</td>
</tr>
</tbody>
</table>

1) *Math 461: Topology* is by far the best class you have ever taken. .......... TRUE | FALSE

2) If $f: X \rightarrow Y$ is a function and $X$ is countable, then $f(X)$ is countable. ...... TRUE FALSE

3) If $f: X \rightarrow Y$ is a function and $Y$ is countable, then $f(X)$ is countable. ...... TRUE FALSE

4) If $f: X \rightarrow Y$ is a function and $Y$ is countable, then $f^{-1}(Y)$ is countable. .... TRUE FALSE

5) If $X$ is a discrete topological space, then any subspace of $X$ is discrete. ............... TRUE FALSE

6) If $X$ is a topological space which is not discrete (i.e., not all subsets are open), then no subspace of $X$ is discrete. ........................................ TRUE FALSE

7) If $X$ is a disconnected topological space, then any subspace of $X$ is disconnected. ................. TRUE FALSE

8) If $X$ is a path-connected topological space, then any subspace of $X$ is path-connected. ........................ TRUE FALSE

9) If $X$ is a compact topological space, then any **closed** subspace of $X$ is compact. .................. TRUE FALSE

10) If $X$ is a Hausdorff topological space, then any subspace of $X$ is Hausdorff. ......................... TRUE FALSE

11) If $X$ is a second-countable topological space, then any subspace of $X$ is second-countable. .......... TRUE FALSE

12) If $X$ is a second-countable topological space, then any **basis** for the topology of $X$ is countable. .................. TRUE FALSE
True or false? (Continued.)

13] If $A$ and $B$ are connected subspaces of a topological space $X$ and $A \cap B \neq \emptyset$, then $A \cup B$ is connected. ....................................................... TRUE FALSE

14] If $A$ and $B$ are connected subspaces of a topological space $X$ and $A \cap B \neq \emptyset$, then $A \cap B$ is connected. ....................................................... TRUE FALSE

15] If $f: X \to Y$ is a continuous function between topological spaces $X$ and $Y$, then for every open subset $U$ of $X$, $f(U)$ is open in $Y$. ......................... TRUE FALSE

16] If $f: X \to Y$ is a homeomorphism between topological spaces $X$ and $Y$, then for every open subset $U$ of $X$, $f(U)$ is open in $Y$. .......................... TRUE FALSE

17] If $f: X \to Y$ is a bijective function between topological spaces $X$ and $Y$, and for every open subset $U$ of $X$, $f(U)$ is open in $Y$, then $f$ is a homeomorphism. TRUE FALSE

18] If $X$ is a Hausdorff space, $Y$ is a compact space, and $f: X \to Y$ is a continuous and bijective function, then $f$ is a homeomorphism. ......................... TRUE FALSE

19] If $X$ and $Y$ are both compact metric spaces, and $f: X \to Y$ is a continuous and bijective function, then $f$ is a homeomorphism. ......................... TRUE FALSE

20] $\mathbb{R}$ and $\mathbb{R}^2$ with the standard topologies are homeomorphic. .................. TRUE FALSE

21] $\mathbb{Z}$ and $\mathbb{Z}^2$ with the discrete topologies are homeomorphic. .................. TRUE FALSE

22] If $f: X \to Y$ is a continuous function between topological spaces $X$ and $Y$, and $X$ is connected and compact, then $f(X)$ is connected and compact. ...... TRUE FALSE

23] If $f: X \to Y$ is a continuous function between topological spaces $X$ and $Y$, and $X$ is separable, then $f(X)$ is separable. ................................. TRUE FALSE

24] If $f: X \to Y$ is a continuous function between topological spaces $X$ and $Y$, and $X$ is Hausdorff, then $f(X)$ is Hausdorff. ................................. TRUE FALSE

25] If $X = \mathbb{R}$ is given the cofinite (also known as finite complement) topology, then the function $f: X \to X$, $f(x) = \sin(x)$, is continuous. .................. TRUE FALSE

26] If $X = \mathbb{R}$ is given the cofinite (also known as finite complement) topology, then the function $f: X \to X$, $f(x) = x^2$, is continuous. .................. TRUE FALSE

39 TOTAL POINTS
II. Fill in the blanks in the following theorem.

**Theorem.** Let $X$ and $Y$ be topological spaces, and let $f : X \to Y$ be a function. Then the following conditions are equivalent:

(i) $f$ is continuous, i.e., for every open subset $U$ of $Y$, $f^{-1}(U)$ is ... open in $X$ ..................;

(ii) for every closed subset $C$ of $Y$, $f^{-1}(C)$ is ... closed in $X$ ..............................;

(iii) for every subset $A$ of $X$, one has ... $f(\bar{A}) \subset \overline{f(A)}$ .................................;

(iv) for every subset $B$ of $Y$, one has ... $f^{-1}(\bar{B}) \supset \overline{f^{-1}(B)}$ ..............................;

(v) for every point $x \in X$ and every neighborhood $V$ of $f(x)$ in $Y$, there is .....................

........ a neighborhood $U$ of $x$ in $X$ such that $U \subset f^{-1}(V)$ (or equivalently $f(U) \subset V$). ........

.................................................................

Prove exactly two implications of your choice from this theorem.

*See the proof of theorem 18.1 in Munkres’ book, pages 104–105.*
III. Consider the following two subspaces of $\mathbb{R}^2$ with the standard topology.

![Diagram showing two subspaces X and Y]

Are $X$ and $Y$ homeomorphic? Justify your answer carefully.

*The spaces $X$ and $Y$ are not homeomorphic.*

Let $p$ be the point of $Y$ drawn in the picture above. Then $Y - \{p\}$ is disconnected, whereas for any point $q \in X$, $X - \{q\}$ is (path-)connected. So if there existed a homeomorphism $f : Y \to X$ then $f$ would induce a homeomorphism between $Y - \{p\}$ and $X - \{f(p)\}$, which is impossible since $X - \{f(p)\}$ is connected but $Y - \{p\}$ is not.
IV. Complete the following definition.

**Definition.** If $X$ is a topological space and $A$ is a subset of $X$, then the *closure* of $A$ in $X$ is

$$\overline{A} = \left\{ x \in X \mid \text{.................. \forall U neighborhood of } x, A \cap U \neq \emptyset \text{ .................} \right\}.$$ 

- If $X = \mathbb{R}^2$ with the **standard** topology and $A = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \text{ and } x \neq 0 \right\}$, then what is $\overline{A}$?

$$\overline{A} = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \right\}.$$ 

- Now let $X$ be a set and let $A$ be a non-empty subset of $X$.

The possible answers for the three following questions are as follows:

1. $\overline{A} = A$.  
2. $\overline{A} = X$.  
3. $\overline{A} = \begin{cases} A & \text{if } A \text{ is finite,} \\ X & \text{if } A \text{ is infinite.} \end{cases}$

Write the number corresponding to the correct answer in each of the boxes below.

- If $X$ has the **indiscrete** topology and $\emptyset \neq A \subset X$, then ........................................ 2

- If $X$ has the **discrete** topology and $\emptyset \neq A \subset X$, then ........................................ 1

- If $X$ has the **cofinite** (also known as finite complement) topology and $\emptyset \neq A \subset X$, then ........................................ 3

**11 TOTAL POINTS**
V. Complete the following two definitions, and then write the precise statement (without proof!) of either the intermediate value theorem or the extreme value theorem.

**Definition.** A topological space $X$ is *disconnected* if and only if \( \exists U, V \subset X \) such that \( U \) and \( V \) are non-empty and open in \( X \), \( U \cup V = X \), and \( U \cap V = \emptyset \).

**Definition.** A topological space $X$ is *compact* if and only if every open cover of \( X \) has a finite subcover, i.e., \( \forall \mathcal{U} \subset \mathcal{P}(X) \), if \( \forall U \in \mathcal{U} \), \( U \) is open in \( X \) and \( \bigcup_{U \in \mathcal{U}} U = X \), then \( \exists n \in \mathbb{N} \) and \( \exists U_1, U_2, \ldots, U_n \in \mathcal{U} \) such that \( U_1 \cup U_2 \cup \cdots \cup U_n = X \).

**Intermediate Value Theorem.** Let \( X \) be a topological space, and let \( f: X \to \mathbb{R} \) be a function.

Assume that \( X \) is *connected*, and that \( f \) is *continuous*.

Then \( \forall a, b \in X \) and \( \forall y \in \mathbb{R} \), if \( f(a) \leq y \leq f(b) \) then \( \exists x \in X \) such that \( f(x) = y \).

**Extreme Value Theorem.** Let \( X \) be a topological space, and let \( f: X \to \mathbb{R} \) be a function.

Assume that \( X \) is *compact and not empty*, and that \( f \) is *continuous*.

Then \( \exists m, M \in X \) such that \( \forall x \in X \), \( f(m) \leq f(x) \leq f(M) \).
VI. Solve only one of the following two problems.

A] Recall that $S^0$ denotes the topological space with only two points \{+1, −1\} and the discrete topology. Prove that a topological space $X$ is disconnected if and only if there exists a continuous and surjective function $f: X \to S^0$.

Assume that $X$ is disconnected. Then $\exists U, V \subset X$ such that $U$ and $V$ are non-empty and open in $X$, $U \cup V = X$, and $U \cap V = \emptyset$. Define $f: X \to S^0$, $f(x) = \begin{cases} -1 & \text{if } x \in U, \\ +1 & \text{if } x \in V. \end{cases}$

Since $U \cup V = X$ and $U \cap V = \emptyset$, $f$ is well-defined. Since $U$ and $V$ are not empty, $f$ is surjective. And since $U$ and $V$ are open, $f$ is continuous.

Conversely, assume that $\exists f: X \to S^0$ continuous and surjective. Define $U = f^{-1}(\{-1\})$ and $V = f^{-1}(\{+1\})$. Since $f$ is continuous, $U$ and $V$ are open in $X$. Since $f$ is surjective, $U$ and $V$ are not empty. And finally we have

$U \cup V = f^{-1}(\{-1\}) \cup f^{-1}(\{+1\}) = f^{-1}(\{-1\} \cup \{+1\}) = f^{-1}(S^0) = X$, and

$U \cap V = f^{-1}(\{-1\}) \cap f^{-1}(\{+1\}) = f^{-1}(\{-1\} \cap \{+1\}) = f^{-1}(\emptyset) = \emptyset$.

B] Recall the following result that we proved in class.

**Lemma.** If $C$ is a compact subset of a Hausdorff space $X$ and $x \in X - C$, then there exist open subsets $U$ and $V$ of $X$ such that $C \subset U$, $x \in V$, and $U \cap V = \emptyset$.

Now let $X$ be a Hausdorff space, and let $C$ and $D$ be compact subsets of $X$ such that $C \cap D = \emptyset$. Prove that there exist open subsets $U$ and $V$ of $X$ such that $C \subset U$, $D \subset V$, and $U \cap V = \emptyset$.

The lemma implies that $\forall x \in D$, $\exists U_x, V_x$ open in $X$ such that $C \subset U_x$, $x \in V_x$, and $U_x \cap V_x = \emptyset$. Then $\{V_x\}_{x \in D}$ is an open cover of $D$, and therefore, since $D$ is compact, $\exists n \in \mathbb{N}, \exists x_1, \ldots, x_n \in D$ such that $V_{x_1} \cup \cdots \cup V_{x_n} \supset D$. Define $V = V_{x_1} \cup \cdots \cup V_{x_n}$ and $U = U_{x_1} \cap \cdots \cap U_{x_n}$. Then $D \subset V$ and $V$ is open in $X$ since it is a union of open sets; $C \subset U$ since $\forall 1 \leq i \leq n$, $C \subset U_{x_i}$, and $U$ is open in $X$ since it is a finite intersection of open sets; and finally $U \cap V = \emptyset$ because if $y \in V$ then $\exists 1 \leq i \leq n$ such that $y \in V_{x_i}$, hence $y \notin U_{x_i}$ since $U_{x_i} \cap V_{x_i} = \emptyset$, and so $y \notin U$.

*Have a great winter break!*