Let \( \mathbb{Z} \) denote the set of all integers, and let \( \mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, \ldots \} \) denote the set of all prime numbers. In this homework assignment you will be guided through a topological proof of the well-known fact that there are infinitely many primes, i.e., that \( \mathbb{P} \) is an infinite set.

**Definition.** For any \( c, d \in \mathbb{Z} \) with \( d > 0 \), let \( AP_{c,d} = \{ c + nd \mid n \in \mathbb{Z} \} \) be the infinite arithmetic progression with initial term \( c \) and common difference \( d \). Define \( \mathcal{B} = \{ AP_{c,d} \mid c, d \in \mathbb{Z}, \ d > 0 \} \).

Prove the following three lemmas.

**Lemma 1.** *The set \( \mathcal{B} \) is a basis for a topology on \( \mathbb{Z} \).*

Let’s consider \( \mathbb{Z} \) together with the topology generated by \( \mathcal{B} \).

**Lemma 2.** *Every non-empty open set of \( \mathbb{Z} \) is infinite.*

**Lemma 3.** *For any \( c, d \in \mathbb{Z} \) with \( d > 0 \), the set \( AP_{c,d} \) is closed.*

Now consider \( X = \mathbb{Z} - \left( \bigcup_{p \in \mathbb{P}} AP_{0,p} \right) \). What exactly is the set \( X \)? Is \( X \) open?

From all this, deduce the following statement.

**Theorem 4.** *There are infinitely many primes.*