On the set $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ define a relation $\sim$ by declaring $(a, b) \sim (c, d)$ if and only if $ad = bc$.

**Problem 1.** Prove that $\sim$ is an equivalence relation.

For all $(a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ define $(a, b) + (c, d) = (ad + bc, bd)$ and $(a, b) \cdot (c, d) = (ac, bd)$.

**Problem 2.** Prove that if $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$, then $(a_1, b_1) + (c_1, d_1) \sim (a_2, b_2) + (c_2, d_2)$ and $(a_1, b_1) \cdot (c_1, d_1) \sim (a_2, b_2) \cdot (c_2, d_2)$.

Let $[a, b]$ denote the equivalence class with respect to $\sim$ of $(a, b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, and define $\mathbb{Q}$ to be the set of equivalence classes of $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$.

For all $[a, b], [c, d] \in \mathbb{Q}$ define $[a, b] + [c, d] = [(a, b) + (c, d)]$ and $[a, b] \cdot [c, d] = [(a, b) \cdot (c, d)]$; these definitions make sense, i.e., they do not depend on the choice of representatives, because of problem 2.

The following properties hold for all $[a, b], [c, d], [e, f] \in \mathbb{Q}$:

(i) $([a, b] + [c, d]) + [e, f] = [a, b] + ([c, d] + [e, f]);$
(ii) $[a, b] + [c, d] = [c, d] + [a, b];$
(iii) $([a, b] \cdot [c, d]) \cdot [e, f] = [a, b] \cdot ([c, d] \cdot [e, f]);$
(iv) $[a, b] \cdot [c, d] = [c, d] \cdot [a, b];$
(v) $([a, b] + [c, d]) \cdot [e, f] = ([a, b] \cdot [e, f]) + ([c, d] \cdot [e, f]);$
(vi) there exists $0 \in \mathbb{Q}$ such that for all $[a, b] \in \mathbb{Q}$, $0 + [a, b] = [a, b];$
(vii) for every $[a, b] \in \mathbb{Q}$ there exists $-[a, b] \in \mathbb{Q}$ such that $-[a, b] + [a, b] = 0;$
(viii) there exists $1 \in \mathbb{Q} - \{0\}$ such that for all $[a, b] \in \mathbb{Q}$, $1 \cdot [a, b] = [a, b];$
(ix) for every $[a, b] \in \mathbb{Q} - \{0\}$ there exists $[a, b]^{-1} \in \mathbb{Q}$ such that $[a, b]^{-1} \cdot [a, b] = 1.$

**Problem 3.** Prove properties (vi), (vii), (viii), and (ix) (notice that in particular you need to explicitly define $0, -[a, b], 1, [a, b]^{-1}$). Prove also at least one of the remaining properties.

**Problem 4.** Among the properties (i)–(ix) above, which ones hold and which ones fail in $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$? Justify your answer.

For $[a, b], [c, d] \in \mathbb{Q}$ define $[a, b] \leq [c, d]$ if and only if $(bd > 0$ and $ad \leq bc)$ or $(bd < 0$ and $ad \geq cb)$.

**Problem 5.** Prove that the above definition of $\leq$ does not depend on the choice of representatives.

**Problem 6.** Prove that the following properties hold for all $[a, b], [c, d], [e, f] \in \mathbb{Q}$:

(a) $[a, b] \leq [c, d]$ or $[c, d] \leq [a, b]$;
(b) if $[a, b] \leq [c, d]$ and $[c, d] \leq [a, b]$, then $[a, b] = [c, d]$;
(c) if $[a, b] \leq [c, d]$ and $[c, d] \leq [e, f]$, then $[a, b] \leq [e, f]$.

**Problem 7.** Define a function $f : \mathbb{Z} \to \mathbb{Q}$ satisfying the following properties:

(A) $f(0) = 0$ and $f(1) = 1$;
(B) for all $m, n \in \mathbb{Z}$, $f(m + n) = f(m) + f(n)$, $f(m \cdot n) = f(m) \cdot f(n)$, and if $m \leq n$ then $f(m) \leq f(n);$  
(C) for all $[a, b] \in \mathbb{Q}$ there exists $n \in \mathbb{N}$ such that $[a, b] \leq f(n)$.

Prove also that $f$ is injective but not surjective.