I. Let $X$, $Y$, and $Z$ be statements.
Are the following statements equivalent to “If $X$ is true, then $Y$ is true or $Z$ is true”?
Please circle your answers.

1] If $X$ is true and $Y$ is false, then $Z$ is true. .................................  YES | NO

2] If $X$ is true and $Z$ is false, then $Y$ is true. .................................  YES | NO

3] If $Y$ is false and $Z$ is false, then $X$ is false. .................................  YES | NO

4] If $Y$ is false or $Z$ is false, then $X$ is false. .................................  YES | NO

5] If $Y$ is true or $Z$ is true, then $X$ is true. .................................  YES | NO

6] If $X$ is true, then $Y$ is true and $Z$ is true. .................................  YES | NO

7] If $X$ is false, then $Y$ is false and $Z$ is false. .................................  YES | NO

8] If $X$ is false, then $Y$ is false or $Z$ is false. .................................  YES | NO

9] $X$ is false or $Y$ is true or $Z$ is true. .................................  YES | NO

10] $X$ is true and $Y$ is true and $Z$ is true. .................................  YES | NO

11] $X$ is true and $Y$ is true, or $X$ is true and $Z$ is true. .................................  YES | NO
II. For each of the following five questions, four possible answers are provided, but only one of them is correct: write the corresponding letter in the box!

II/1. Let $f: X \rightarrow Y$ be a function. Let $x$ and $x'$ be elements of $X$ such that $f(x) = f(x')$.

What do we need to know about $f$ to conclude that $x = x'$?

A] Nothing: this is true for all functions $f$.
B] We need $f$ to be injective.
C] We need $f$ to be surjective.
D] We need $f$ to be bijective.

II/2. Let $f: X \rightarrow Y$ be a function. Let $x$ and $x'$ be elements of $X$ such that $x = x'$.

What do we need to know about $f$ to conclude that $f(x) = f(x')$?

A] Nothing: this is true for all functions $f$.
B] We need $f$ to be injective.
C] We need $f$ to be surjective.
D] We need $f$ to be bijective.

II/3. Let $f: X \rightarrow Y$ be a function. Let $y$ be an element of $Y$.

What do we need to know about $f$ to conclude that $y = f(x)$ for some $x \in X$?

A] Nothing: this is true for all functions $f$.
B] We need $f$ to be injective.
C] We need $f$ to be surjective.
D] We need $f$ to be bijective.

II/4. Let $f: X \rightarrow Y$ be a function. Let $y$ be an element of $Y$.

What do we need to know about $f$ to conclude that $y = f(x)$ for exactly one $x \in X$?

A] Nothing: this is true for all functions $f$.
B] We need $f$ to be injective.
C] We need $f$ to be surjective.
D] We need $f$ to be bijective.

II/5. Let $f: X \rightarrow Y$ be a function. Let $y$ be an element of $Y$.

What do we need to know about $f$ to conclude that $y = f(x)$ for at most one $x \in X$?

A] Nothing: this is true for all functions $f$.
B] We need $f$ to be injective.
C] We need $f$ to be surjective.
D] We need $f$ to be bijective.
III. Let \( a \) and \( b \) be integers, i.e., \( a, b \in \mathbb{Z} \).

III/1. What exactly does it mean to say that “\( a \) is divisible by \( b \)”, or equivalently that “\( b \) divides \( a \)”?

III/2. Is it true or false that for every natural number \( n \in \mathbb{N} \), \( 6^n \) is not divisible by 5? ……
Prove your claim.
IV. Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of real numbers.

IV/1. What exactly does it mean to say that \((x_n)_{n \in \mathbb{N}}\) is convergent?

IV/2. Prove the following statement: If \((x_n)_{n \in \mathbb{N}}\) is convergent, then for every \(\varepsilon \in \mathbb{R}_{>0}\) there exists an \(N \in \mathbb{N}\) such that for all \(m, n \in \mathbb{N}\), if \(m \geq N\) and \(n \geq N\) then \(|x_m - x_n| < \varepsilon\).
IV/3. What is the contrapositive of the statement in IV/2?

IV/4. Use IV/3 to show that the sequence \( x_n = (-1)^n \) is divergent.
V. For each natural number $n \in \mathbb{N}$, define $x_n = \sum_{j=1}^{n} \frac{1}{j^2}$.

V/1. Prove that for all $n \in \mathbb{N}$, $x_n \leq 2 - \frac{1}{n}$.

V/2. Does the sequence $(x_n)_{n \in \mathbb{N}}$ defined above converge in $\mathbb{R}$? Prove your claim.
VI. VI/1. Define a relation $\sim$ on the set of real numbers $\mathbb{R}$ as follows: for all $x, y \in \mathbb{R}$, declare $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Is $\sim$ an equivalence relation? Prove your claim.

VI/2. More generally, let $A$ be a subset of $\mathbb{R}$ and define a relation $\sim$ on $\mathbb{R}$ by declaring $x \sim y$ if and only if $x - y \in A$. What conditions must $A$ satisfy in order for $\sim$ to be an equivalence relation?
VII. Suppose that you have a set $A$ and a subset $B \subseteq A$ such that $B \neq A$.

VII/1. What exactly do these conditions mean?

$B \subseteq A$

$B \neq A$

VII/2. Given $A$ and $B$ satisfying the above conditions, is it possible for $A$ and $B$ to have the same cardinality, i.e., $A \simeq B$?  

A] No, it is not possible for any $A$.
B] Yes, it is possible for any $A$.
C] Yes, but only if $A$ is empty.
D] Yes, but only if $A$ is not empty.
E] Yes, but only if $A$ is finite.
F] Yes, but only if $A$ is infinite.
G] Yes, but only if $A$ is countable.
H] Yes, but only if $A$ is uncountable.

VII/3. Prove that your answer to VII/2 is correct.