Math 520A  Problem Set 2  Spring 2010

We first study some subgroups of $\text{Gl}_2(\mathbb{Z}_3)$.

1. Let
   \[ H = \{ \pm I_2 \} \cup \{ A \in \text{Gl}_2(\mathbb{Z}_3) : \det(A) = 1, \ \text{tr}(A) = 0 \} \].
   a) What are the orders of the elements of $H$?
   b) Construct a 1-1 homomorphism $f : Q_8 \to \text{Gl}_2(\mathbb{Z}_3)$ with image $H$, where $Q_8$ is the quaternionic group of order 8. Deduce that $H$ is a subgroup of $\text{Gl}_2(\mathbb{Z}_3)$ isomorphic to $Q_8$.
   c) Show that $H \triangleleft \text{Gl}_2(\mathbb{Z}_3)$.
   d) Show the group $\text{Gl}_2(\mathbb{Z}_3)/H$ is not abelian.

2. Find a subgroup $L$ of $\text{Gl}_2(\mathbb{Z}_3)$ of order 16 containing $H$. What are the orders of its elements?

3. Show that $D_8$ embeds in $L$.

Now we study subgroups of $S_4$.

4. Let
   \[ H = \{ e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3) \} \subset A_4 \]
   a) Show that $H \triangleleft S_4$.
   b) Show that $S_4/H \cong S_3$.
   c) Find a subgroup $K$ of order 8 in $S_4$ containing $H$, and construct an isomorphism from a familiar group of order 8 to $K$.

5. Show that $A_4$ has no subgroup of order 6. Deduce that $A_4$ is the only subgroup of $S_4$ of order 12.