Problem Set 1  Math 520A  Spring 2016

We first study some subgroups of GL$_2$(Z$_3$).

1. Let $H = \{ \pm I_2 \} \cup \{ A \in \text{GL}_2(Z_3) : \det(A) = 1, \ \text{tr}(A) = 0 \}$.
   (a) What are the orders of the elements of $H$?
   (b) Construct a 1-1 homomorphism $f : Q_8 \rightarrow \text{GL}_2(Z_3)$ with image $H$, where $Q_8$ is the quaternionic group of order 8. Deduce that $H$ is a subgroup of $\text{GL}_2(Z_3)$ isomorphic to $Q_8$.
   (c) Show that $H \triangleleft \text{GL}_2(Z_3)$.
   (d) Show the group $\text{GL}_2(Z_3)/H$ is not abelian.

2. Find a subgroup $L$ of $\text{GL}_2(Z_3)$ of order 16 containing $H$. What are the orders of its elements? (Hint: find a subgroup $K$ of order 2, not contained in $H$, such that $K \subset N_G(H)$. Apply the second Noether theorem.)

3. Show that $D_8$ embeds in $L$.

4. Show that $Z_8$ embeds in $L$.

5. Use rational canonical form to find all the conjugacy classes in $\text{GL}_2(Z_3)$. Specify the order of each.

6. Are there elements of $\text{SL}_2(Z_3)$ that are conjugate in $\text{GL}_2(Z_3)$ but not in $\text{SL}_2(Z_3)$?

Now we study subgroups of $S_4$.

7. Let $H = \{ e, (1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4), (1 \ 4)(2 \ 3) \} \subset A_4$
   (a) Show that $H \vartriangleleft S_4$.
   (b) Show that $S_4/H \cong S_3$. (Hint: use the second Noether theorem.)
   (c) Find a subgroup $K$ of order 8 in $S_4$ containing $H$ (argue as in problem 2, above), and construct an isomorphism from a familiar group of order 8 to $K$.

8. (a) Show that $A_4$ has no subgroup of order 6.
   (b) Deduce that $A_4$ is the only subgroup of $S_4$ of order 12. (Hint: if $K \subset A_4$ has order 6, consider the composite $K \subset A_4 \xrightarrow{\pi} A_4/H$ to show that $K$ cannot exist. Now, if $L \subset S_4$ has order 12, study the composite $L \subset S_4 \xrightarrow{\pi} S_4/A_4$ to show $L = A_4$.)