Recall that if $F$ is a field, then $A, B \in \text{Gl}_n(F)$ are similar (conjugate) if and only if they have the same rational canonical form. Recall also that if $f \in F[x]$, then $f(A) = 0$ if and only if the minimal polynomial of $A$ divides $f$. In particular, $A^k = I_n$ if and only if the minimal polynomial of $A$ divides $x^k - 1$. (Of course, the prime decomposition of $x^k - 1$ in $F[x]$ depends very much on the field $F$.)

In particular, we want to study all rational canonical forms for matrices whose order divides $k$ and to determine the precise order in each case. To do this, we’ll need to find the order of the companion matrix $C(f)$ for every monic polynomial $f$ dividing $x^k - 1$ in $F[x]$. Write $B = C(f)$. we have that $B^k = I$, hence the order of $B$ divides $k$. Since the minimal polynomial of $C(f)$ is $f$, the order of $B$ will be the smallest positive divisor, $\ell$, of $k$ such that $f$ divides $x^\ell - 1$.

1. Give representatives for all the distinct conjugacy classes of elements of order 7 in $\text{Gl}_3(\mathbb{Z}_2)$.
2. Give representatives for all the distinct conjugacy classes of elements of order 7 in $\text{Gl}_6(\mathbb{Z}_2)$.
3. Let $A$ be your favorite element of order 7 in $\text{Gl}_3(\mathbb{Z}_2)$. Find the minimal polynomials of all powers of $A$. Which powers of $A$ are conjugate to $A$?
4. Give representatives for all the distinct conjugacy classes of elements of in $\text{Gl}_3(\mathbb{Z}_2)$ whose order is a power of 2. What is the order in each case?
5. Give representatives for all the distinct conjugacy classes of elements of in $\text{Gl}_6(\mathbb{Z}_2)$ whose order is a power of 2. What is the order in each case?
6. What is the smallest value of $n$ such that $\text{Gl}_n(\mathbb{Z}_2)$ has an element of order 6? Give representatives for all conjugacy classes of order 6 for that value of $n$.
7. In $\mathbb{Z}_3[x]$, we have

\[x^4 + 1 = (x^2 + x - 1)(x^2 - x - 1)\]

Use this to find representatives of all the conjugacy classes in $\text{Gl}_2(\mathbb{Z}_3)$ of elements whose order divides 8. Give the order in each case.