Here are some calculations we may use below:

\[
R\left( e_1, \frac{\pi}{2} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}
\]

\[
R\left( e_2, \frac{\pi}{2} \right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}
\]

\[
R\left( e_3, \frac{\pi}{2} \right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Recall that the linear isometries of \( \mathbb{R}^3 \) are induced by \( 3 \times 3 \) orthogonal matrices \( A \), i.e., \( 3 \times 3 \) matrices \( A = [u|v|w] \) whose columns, \( u, v, w \) form an orthonormal basis of \( \mathbb{R}^3 \). The set of such matrices is the orthogonal group \( O_3 \).

Recall that for a square matrix \( A \), the eigenspace of \( (A, 1) \) is the set of fixed-points of the linear transformation, \( T_A \), induced by \( A \). Similarly, the eigenspace of \( (A, -1) \) is the set of vectors with \( Av = -v \).

Finally, recall that the linear isometries of \( \mathbb{R}^3 \) all have one of the following standard forms:

- The identity matrix \( I_3 \).
- Rotations \( \rho_{(u, \theta)} \) with \( u \) a unit vector and \( \theta \in (0, 2\pi) \). Here, the only ambiguity is that \( \rho_{(u, \theta)} = \rho_{(-u, -\theta)} \).
- The reflection \( \sigma_V \) of \( \mathbb{R}^3 \) across the 2-dimensional linear subspace \( V \). It has the following formula: if \( u \) is a unit normal to the plane \( V \), then

\[
\sigma_V(x) = x - 2\langle x, u \rangle u
\]

for all \( x \in \mathbb{R}^3 \).
- Rotation-reflections \( \sigma_V \rho(u, \theta) \) where \( u \) is a unit vector in \( V^\perp \) and \( \theta \in (0, 2\pi) \setminus \{\pi\} \). Here, the ambiguity is the same as for the rotations.
- \( -I_3 \).

1. Let \( A \in O_3 \). If the eigenspace of \( (A, 1) \) has dimension 2, what kind of transformation is \( T_A \), a rotation, a reflection or a rotation-reflection? If it’s a rotation, what is \( u \)? If it’s a reflection, what is \( V \)?

2. Let \( A \in O_3 \). If the eigenspace of \( (A, 1) \) has dimension 1, what kind of transformation is \( T_A \), a rotation, a reflection or a rotation-reflection? If it’s a rotation, what is \( u \)? If it’s a reflection, what is \( V \)?
3. Let $V = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right)$.

(a) Find a unit vector in $V^\perp$. Test that it’s orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ and has unit length.

(b) Use formula (1) to obtain the matrix for $\sigma_V$. Hint: It’s an isometry of the cube.

4. We study the transformation $\alpha = \rho(e_2, \pi/2)\rho\left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{2\pi}{3}\right)$.

(a) What matrix induces $\alpha$?

(b) Find a unit vector $u$ with $\alpha(u) = u$.

(c) Find the value of $\theta$ for which $\alpha = \rho(u, \theta)$. 