Exam 2  
Fall 2013

Math 432-532

1. Let $P = e_1$, $Q = \cos \frac{2\pi}{3} e_1 + \sin \frac{2\pi}{3} e_2$, $R = \cos \frac{4\pi}{3} e_1 + \sin \frac{4\pi}{3} e_2$, and $N = e_3$.

Let
\[ u(t) = \cos t N + \sin t P, \]
\[ v(t) = \cos t N + \sin t Q, \]
\[ w(t) = \cos t N + \sin t R, \]
with $0 < t < \frac{\pi}{2}$. Let
\[ \Delta(t) = \Delta u(t)v(t)w(t), \]
the spherical triangle whose vertices are $u(t)$, $v(t)$ and $w(t)$.

(a) Show that $\Delta(t)$ is an equilateral triangle.
(b) Use a computer algebra system to compute the angle sum in $\Delta(t)$.
(c) Show that as $t$ varies, the angle sum can take on any value in $(\pi, 3\pi)$.
(d) What is the side length of $\Delta(t)$?

2. Find the vertices of an inscribed regular tetrahedron in $S^2$ one of whose vertices is $e_3$. Call them $e_3, v_1, v_2, v_3$ (note the connection with problem 1).

(a) What are the angles in the spherical equilateral triangles formed by the vertices of this tetrahedron?
(b) Find the matrix for $\rho_{(e_3, \frac{2\pi}{3})}$ with respect to the standard basis and show it permutes the vertices $v_1, v_2, v_3$.
(c) Find the matrix for $\rho_{(v_1, \frac{2\pi}{3})}$ with respect to the standard basis and show it permutes the vertices $e_3, v_2, v_3$.
(d) Use these matrices to compute the composite
\[ \rho_{(e_3, \frac{2\pi}{3})} \circ \rho_{(v_1, \frac{2\pi}{3})}. \]
What rotation is the composite? Write it in the form $\rho_{(w, \theta)}$ for specific $w$ and $\theta$.\(^1\)

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\(^1\)Note that rotational angles are signed angles, unlike the angles in a triangle. The sign comes from the choice of unit normal to the plane of rotation. If you choose the opposite unit normal, the angle changes sign. In particular, you can’t
(e) Let $\sigma_\ell$ be the reflection that interchanges $e_3$ and $v_3$. Find the matrix of $\sigma_\ell$ and show it permutes the vertices $v_1$, $v_2$.

(f) Let $\sigma_m$ be the reflection that interchanges $e_3$ and $v_2$. Find the matrix of $\sigma_m$ and show it permutes the vertices $v_1$, $v_3$.

(g) Find the matrix of the composite $\sigma_\ell \circ \sigma_m$. What rotation does it represent? Write it in the form $\rho(w, \theta)$ for specific $w$ and $\theta$.

(h) Let $\beta = \rho(e_3, \frac{2\pi}{3}) \circ \sigma_\ell$ with $\ell$ as above.
   (i) Show that $\beta$ is a rotation-reflection. Write it in the standard form $\beta = \rho(w, \theta) \circ \sigma_n$, where $n$ is the spherical line with pole $w$. (I.e., please specify $w$ and $\theta$. $Z$ is then implicitly defined.)
   (ii) Calculate the effect of $\beta$ on the vertices $e_3$, $v_1$, $v_2$, $v_3$.
   (iii) What rotation is $\beta^2$? (Write it in the form $\rho(u, \phi)$ for specific $u$ and $\phi$.) What is $\beta^4$?

\[\text{calculate the angle by a simple formula involving } \cos^{-1}. \text{ You need to know the sine of the angle, as well, i.e., to know whether the sine is positive or negative.}\]