Euclidean geometry is the geometry of the visible plane and the properties of its lines, circles, polygons, etc. The Greeks, as codified by Euclid, set up axioms that led to formal derivations of the visible properties of these shapes.

Today we derive the same properties from calculus and analytic geometry. Instead of using Euclid’s axioms, we use the axioms of set theory and of the real line, obtaining much more information about these visible shapes than was available from Euclid’s axioms. For instance, we can derive the very powerful laws of cosines and of sines, taught today in high school.

So one way of thinking about analytic geometry is that it is a realization of Euclid’s axioms: a system that satisfies Euclid’s axioms and much more. A realization that is fundamental to the foundations of modern mathematics.

From the very beginning in Greek times, there was controversy about Euclid’s parallel postulate: that given a line \( \ell \) and a point \( P \) not on it there is a unique line \( m \) through \( P \) parallel to \( \ell \). From the standpoint of the visible or naive geometry of the plane it seems clearly true, and many thought it could be derived from Euclid’s other four axioms.

Of course, in analytic geometry, the parallel postulate is an easy theorem: a line \( m \) is parallel to \( \ell \) if and only if they have the same slope. And the point-slope formula shows that there is a unique line \( m \) through \( P \) with a prescribed slope.

Let us now segue to the study of non-Euclidean geometry. In the 19th century, Bolyai, Lobachevsky and others showed that there is a viable axiomatic system of geometry based on the first four of Euclid’s axioms (those other than the parallel postulate) together with the assumption that given a line \( \ell \) and a point \( P \) not on it, there is more than one line through \( P \) parallel to \( \ell \).
Poincaré was able to provide a realization for the non-Euclidean axioms using analytic tools analogous to those used in “analytic geometry”. The tools in question are called Riemannian geometry. In fact, Poincaré’s realization of non-Euclidean geometry comes from putting a nonstandard Riemannian structure on the plane! The resulting geometric structure is called hyperbolic space.

Another geometry of obvious importance to us is spherical geometry: the geometry of great circles on the sphere. The key property of great circles is that they are curves that realize shortest distances between points (as calculated via the arc length formula in calculus). Because they are distance-minimizing curves (or geodesics, in the language of Riemannian geometry) they generate the geometry of importance for things like plotting travel routes by airplanes. Given three cities, the spherical “triangle” that joins them is obtained from the great circles joining each pair of points.

Spherical geometry can be studied using the standard geometric structure of 3-dimensional space, i.e., the standard tools in the calculus and linear algebra of several variables. No new structure need be introduced.

Note that spherical geometry is not planar: it is realized on the sphere and not the plane.

The three geometries discussed above will be derived and studied using analytic tools. The basic properties of triangles in each context will be studied. An interesting fact is that while the angles in a Euclidean triangle always add up to $\pi$ radians, the angles in a spherical triangle can add up to any number in the open interval $(\pi, 3\pi)$. In hyperbolic space, the angle sum for a triangle is always less than $\pi$.

This fact about angle sums in the three geometries is related to a concept in Riemannian geometry known as curvature. The curvature of Euclidean space is 0 at each point. The curvature of the sphere is 1 at each point. The curvature of hyperbolic space is $-1$ at each point. We will not discuss curvature in this class, but it is a useful concept.

A very important concept that will be developed in all three geometries is the group of isometries, or distance-preserving transformations, of each of these spaces. Isometries are the what underlies the notion of congruence of triangles and other figures. Two triangles are congruent if there is an isometry carrying the one triangle onto the other. The isometries of the Euclidean plane are studied in greater detail in Math 331 and Math 531.

The textbook is available for free online. It will be updated periodically during the term, so it does not make sense to print out more than you need at the moment. But it could be very useful to save it on your computer, tablet, smart phone, etc.

There will be three take-home exams for this class, one each on Euclidean, spherical and hyperbolic geometry. We will also have graded group work every class period. The point count for the final grade is as follows.
In-class work 13%
Each take-home exam 29%

You are strongly encouraged to discuss this material with each other and with me, both in office hours and in class. Verbalizing mathematical questions is a very useful step toward understanding them. Classroom discussion is strongly encouraged. Please ask questions! If there is something you don’t understand or can’t follow, there will be a number of other people in the class in the same boat. So a number of people will benefit if you ask.

My goal here is to teach you, not to penalize you. The test for all of us is how you do on the exams. So please make use of the class and office hours to get my help. I am happy to give it.

It is very important to stay current with the material. If you fall behind, it will be hard to catch up. If you are having trouble, please do come to office hours early on. If you leave it until the last minute, you probably won’t be able to learn it in time.

The ultimate test is being able to solve problems. Keep your curiosity alive and follow it where it leads.