1. Suppose \( V \) is the invariant direct sum of \( W \) and \( Z \), so that the map
\[
\iota : W \oplus Z \to V
\]
\[
(w, z) \mapsto w + z
\]
is an isomorphism. Let \( \mathcal{B} = v_1, \ldots, v_n \) be a basis of \( V \) such that \( v_1, \ldots, v_k \) is a basis of \( W \). It does not then follow that \( v_{k+1}, \ldots, v_n \) is a basis of \( Z \).

For instance, let \( A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \) and \( T = T_A \). Then we may take \( W = \text{Span}(e_1) \) and \( Z = \text{Span}(e_2) \). But we could choose \( \mathcal{B} = v_1, v_2 \) with \( v_1 = e_1, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). Then \( v_1 \) is in fact a basis of \( W \), but \( v_2 \) does not lie in \( Z \), and, in fact, \( \text{Span}(v_2) \) is not \( T \)-invariant. Note that for this choice of \( T \) and this choice of \( \mathcal{B} \), we have
\[
[T]_\mathcal{B} = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix},
\]
so this is not a correct basis to obtain the desired form from our matrix.

What you need to show here is that if \( V \) is the invariant direct sum of \( W \) and \( Z \) and if \( w_1, \ldots, w_k \) and \( z_1, \ldots, z_\ell \) of \( W \) and \( Z \), respectively, then \( w_1, \ldots, w_k, z_1, \ldots, z_\ell \) is a basis of \( V \).

2. On 1b), it is tempting to find a right inverse \( B \) for \( A \) (i.e., \( AB = I_n \)), which exists as \( T_A \) is onto, and then deduce \( A \) is invertible. The common argument for that deduction would use Gauss elimination. But our assumption here is that \( R \) is a commutative ring, not a field. So Gauss elimination can't be used.

The argument to use here is as follows: Since \( T_A \) is an isomorphism, its inverse function, \( T_A^{-1} \) is linear, and hence has the form \( T_B \) for some \( n \times n \) matrix \( B \). We get
\[
I_n = T_B T_A = T_{BA} \quad \text{so} \quad BA = I_n.
\]
\[
I_n = T_A T_B = T_{AB} \quad \text{so} \quad AB = I_n.
\]