Math 424-524 Notes on Problem Set 1 Fall 2007

1. Suppose V is the invariant direct sum of W and Z, so that the map
$$\iota : W \oplus Z \to V$$
$$(w, z) \mapsto w + z$$
is an isomorphism. Let $\mathcal{B} = v_1, \dots v_n$ be a basis of V such that $v_1, \dots, v_k$ is a basis of W. It does *not* then follow that $v_{k+1}, \dots, v_n$ is a basis of Z.

   For instance, let $A = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right]$ and $T = T_A$, Then we may take $W = \text{Span}(e_1)$ and $Z = \text{Span}(e_2)$. But we could choose $\mathcal{B} = v_1, v_2$ with $v_1 = e_1$, $v_2 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. Then v_1 is in fact a basis of W, but v_2 does not lie in Z, and, in fact, $\text{Span}(v_2)$ is not T-invariant. Note that for this choice of T and this choice of $\mathcal{B}$, we have
$$[T]_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix},$$
so this is not a correct basis to obtain the desired form from our matrix.

   What you need to show here is that if V is the invariant direct sum of W and Z and if $w_1, \dots, w_k$ and $z_1, \dots, z_\ell$ of W and Z, respectively, then $w_1, \dots, w_k, z_1, \dots, z_\ell$ is a basis of V.

2. On 1b), it is tempting to find a right inverse B for A (i.e., $AB = I_n$), which exists as T_A is onto, and then deduce A is invertible. The common argument for that deduction would use Gauss elimination. But our assumption here is that R is a commutative ring, not a field. So Gauss elimination can't be used.

   The argument to use here is as follows: Since T_A is an isomorphism, its inverse function, T_A^{-1} is linear, and hence has the form T_B for some $n \times n$ matrix B. We get
$$I_n = T_B T_A = T_{BA} \quad \text{so } BA = I_n.$$
$$I_n = T_A T_B = T_{AB} \quad \text{so } AB = I_n.$$