1. Let $R$ be a commutative ring. A matrix $A \in M_n(R)$ is said to be invertible if there is a matrix $B \in M_n(R)$ with $AB = BA = I_n$.

a) Show that if $B$ and $C$ are both inverses of an invertible matrix $A$, i.e., if $AB = BA = I_n$ and $AC = CA = I_n$, then $B = C$.

Thus, if $A$ is invertible, we may unambiguously write $A^{-1}$ for its inverse.

b) Show that $A$ is invertible $\iff$ the transformation $T_A : R^n \to R^n$ induced by $A$ is an isomorphism.

c) Show that $A$ is invertible $\iff$ the columns of $A$ form a basis of $R^n$.

2. Let $F$ be a field. Two matrices $A, B \in M_n(F)$ are said to be similar (written $A \sim B$) if there is an invertible matrix $P \in M_n(F)$ with $B = P^{-1}AP$. Let $V$ be an $n$-dimensional vector space over $F$ and let $T : V \to V$ be linear. Let $B$ be a basis of $V$ and let $A = [T]_B$. Show that a matrix $B \in M_n(F)$ is similar to $A$ if and only if there is a basis $B'$ of $V$ with $B = [T]_{B'}$.

3. Let $V$ be a vector space over $F$ and let $T : V \to V$ be linear. A subspace $W$ of $V$ is said to be $T$-invariant if $T(w) \in W$ for all $w \in W$. If $W$ is $T$-invariant, we write $T_W : W \to W$ for the restriction of $T$ to $W$: $T_W(w) = T(w)$.

a) Show that the following conditions are equivalent:

(i) $W$ is $T$-invariant.

(ii) For any basis $B = v_1, \ldots, v_n$ of $V$ such that $B' = v_1, \ldots, v_k$ is a basis of $W$, the matrix $[T]_B$ has the form

\[
[T]_B = \begin{bmatrix}
A & X \\
0 & B
\end{bmatrix}
\]

where $A$ is $k \times k$, $X$ is $k \times (n - k)$, $0$ is the $(n - k) \times k$ zero-matrix, and $B$ is $(n - k) \times (n - k)$.

(iii) There exists a basis $B = v_1, \ldots, v_n$ of $V$ such that $B' = v_1, \ldots, v_k$ is a basis of $W$, and the matrix $[T]_B$ has the form

\[
[T]_B = \begin{bmatrix}
A & X \\
0 & B
\end{bmatrix}
\]

where $A$ is $k \times k$, $X$ is $k \times (n - k)$, $0$ is the $(n - k) \times k$ zero-matrix, and $B$ is $(n - k) \times (n - k)$.

b) If the condition (ii) or (iii) holds, show that $A = [T_W]_{B'}$. 
4. Let $T : V \to V$ be a linear transformation of a vector space over $F$. Show that a one-dimensional subspace $\text{Span}(v), v \neq 0,$ is $T$-invariant if and only if there is an element $a \in F$ such that $T(v) = av$ (i.e., if and only if $v$ is an eigenvector of $T$ with eigenvalue $a$).

5. Let $T : V \to V$ be a linear transformation of a vector space over $F$ and let $W$ be a $T$-invariant subspace. A $T$-invariant complement of $W$ is a $T$-invariant subspace, $Z$, of $V$ such that the linear transformation

$$\iota : W \oplus Z \to V$$

given by $\iota(w, z) = w + z$ is an isomorphism. Here $W \oplus Z$ is the set of ordered pairs $(w, z)$ with $w \in W$ and $z \in Z$ and is a vector space via

$$(w, z) + (w', z') = (w + w', z + z')$$

$$a(w, z) = (aw, az)$$

for $w, w' \in W$, $z, z' \in Z$ and $a \in F$.

If $W$ admits a $T$-invariant complement, we say $W$ is a $T$-invariant direct summand of $V$.

Show that $W$ is a $T$-invariant direct summand of $V$ if and only if there is a basis $\mathcal{B} = v_1, \ldots, v_n$ of $V$ such that $\mathcal{B}' = v_1, \ldots, v_k$ is a basis of $W$, and the matrix $[T]_\mathcal{B}$ has the form

$$[T]_\mathcal{B} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

as in (1) with $X$ replaced by the appropriate zero-matrix.