Math 424-524 Midterm Solutions Fall '97

1. Let $B = \left[\begin{smallmatrix} 1 & 3 \\ 2 & 6 \end{smallmatrix}\right]$. Define $T : \mathbf{Q}^{2\times 2} \to \mathbf{Q}^{2\times 2}$ by $T(A) = BA$. Give bases for the kernel and range of T.

   Solution: Conceptually, $BA = 0$ if the range of A is contained in the kernel of B. B reduces to $\left[\begin{smallmatrix} 1 & 3 \\ 0 & 0 \end{smallmatrix}\right]$, so the kernel of B is $\mathrm{Span}(\left[\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right])$.

   The range of A is the span of its columns. Thus, A is in $\ker T$ if each of its columns is in $\mathrm{Span}(\left[\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right])$, i.e., if $A = \begin{bmatrix} a\cdot(-3) & b\cdot(-3) \\ a\cdot(1) & b\cdot(1) \end{bmatrix}$ for some $a, b \in \mathbf{Q}$.

   We can also derive this directly, without discussing the range of A and the kernel of B: If $A = \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]$, then

$$BA = \begin{bmatrix} x+3z & y+3w \\ 2(x+3z) & 2(y+3w) \end{bmatrix},$$

   so $A \in \ker T$ if and only if $x + 3z = 0$ and $y + 3w = 0$, i.e., if and only if $x = -3z$ and $y = -3w$. But then $\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] = z \cdot \left[\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} y \\ w \end{smallmatrix}\right] = w \cdot \left[\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right]$.

   Thus, the elements of $\ker T$ are linear combinations of $A_1 = \left[\begin{smallmatrix} -3 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix} 0 & -3 \\ 0 & 1 \end{smallmatrix}\right]$. The matrices A_1 and A_2 are linearly independent, as $aA_1 + bA_2 = \begin{bmatrix} a\cdot(-3) & b\cdot(-3) \\ a\cdot(1) & b\cdot(1) \end{bmatrix}$. If this is 0, then $-3a = 0$ and $-3b = 0$, so $a = b = 0$.

   Thus, A_1 and A_2 form a basis for $\ker T$.

   For the range of T, we can again use the calculation that if $A = \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]$, then

$$\begin{aligned} BA &= \begin{bmatrix} x+3z & y+3w \\ 2(x+3z) & 2(y+3w) \end{bmatrix} \\ &= \begin{bmatrix} a\cdot 1 & b\cdot 1 \\ a\cdot 2 & b\cdot 2 \end{bmatrix}, \end{aligned}$$

   where $a = x+3z$ and $b = y+3w$. Since x, y, z, w are arbitrary, so are a and b, so the range of T is $\mathrm{Span}(\left[\begin{smallmatrix} 1 & 0 \\ 2 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 2 \end{smallmatrix}\right])$. As above, these two matrices are linearly independent, and hence form a basis for the range of T.

2. Let $t_1, \dots, t_k \in F$ be distinct. Let $p(x) = (x - t_1) \dots (x - t_k)$. Let $A \in F^{n \times n}$ with $p(A) = 0$, i.e., $(A - t_1 I_n) \dots (A - t_k I_n) = 0$.

   Let $P_1(x), \dots, P_k(x)$ be the Lagrange polynomials for $t_1, \dots, t_k$, i.e.,

$$P_i(x) = \frac{\prod_{j \neq i}(x - t_j)}{\prod_{j \neq i}(t_i - t_j)}.$$

   Let $B_i = P_i(A) = \frac{1}{\prod_{j \neq i}(t_i - t_j)} \prod_{j \neq i}(A - t_j I_n)$. Show the following:

   a) $B_1 + \dots + B_k = I_n$.
   b) $B_i B_j = 0$ for $i \neq j$.
   c) $B_i^2 = B_i$ for all i.
   d) $t_1 B_1 + \dots + t_k B_k = A$.

   (Hint: What are $P_1(x) + \dots + P_k(x)$ and $t_1 P_1(x) + \dots + t_k P_k(x)$?)

   Solution: We first solve the hint. Recall that if f is a polynomial of degree less than k, then formula (4-14) in section 4.3 gives:

$$f = f(t_1)P_1 + \dots + f(t_k)P_k.$$

   Apply this first to $f = 1$. Since f is a the constant function whose value is 1 on every point, we see that $1 = P_1 + \dots + P_k$. Now $1(A) = I_n$, so

$$I_n = P_1(A) + \dots + P_k(A) = B_1 + \dots + B_k,$$

   so a) is true.

   Next, we apply the same formula to $f = x$. Evaluating x at t_i gives t_i, so $x = t_1 P_1 + \dots + t_k P_k$. Evaluating these polynomials at A gives

$$A = t_1 B_1 + \dots + t_k B_k,$$

   so d) is true.

   We now prove b). Note first that $\prod_{k \neq i}(x - t_k)$ divides P_i. For the same reason, $(x - t_i)$ divides P_j if $i \neq j$. Thus $(x - t_1) \dots (x - t_k)$ divides $P_i P_j$. Thus $0 = (A - t_1 I) \dots (A - t_k I)$ divides $P_i(A) P_j(A) = B_i B_j$, so $B_i B_j = 0$.

   To obtain c), multiply both sides of a) on the left by B_i:

$$B_i B_1 + \dots + B_i B_k = B_i.$$

   For $i \neq j$, $B_i B_j = 0$, so the left hand side is just B_i^2, giving c).

3. Let $A \in F^{n\times n}$ with $A^2 = A$. Show that A is similar to $\left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right]$ for some $k \le n$.

   Solution: We first show the following:

   Claim 1. *If α is in the range of A, $A\alpha = \alpha$.*

   Proof. Since α is in the range of A, $\alpha = A\beta$ for some β. Thus $A\alpha = A \cdot A\beta = A^2\beta = A\beta = \alpha$, since $A^2 = A$. □

   Now let $\alpha_1, \ldots, \alpha_k$ be a basis for Range(A). Since

$$\dim \operatorname{Range}(A) + \dim \ker(A) = n,$$

   the kernel of A has dimension $n - k$. Let $\alpha_{k+1}, \ldots, \alpha_n$ be a basis for $\ker(A)$.

   Claim 2. *With the choices above, $\mathfrak{B} = \alpha_1, \ldots, \alpha_n$ is a basis for F^n.*

   Proof. Since F^n has dimension n, it suffices to show that $\alpha_1, \ldots, \alpha_n$ are linearly independent. Suppose that

$$a_1\alpha_1 + \cdots + a_n\alpha_n = 0. \tag{1}$$

   Then multiplication by A gives

$$a_1A\alpha_1 + \cdots + a_nA\alpha_n = 0. \tag{2}$$

   For $i > k$, $\alpha_i \in \ker(A)$, so $A\alpha_i = 0$. For $i \le k$, $A\alpha_i = \alpha_i$ by Claim 1. Thus, (2) reduces to

$$a_1\alpha_1 + \cdots + a_k\alpha_k = 0. \tag{3}$$

   But $\alpha_1, \ldots, \alpha_k$ is a basis of Range(A), and hence is linearly independent. Thus $a_i = 0$ for $i \le k$. But then (1) reduces to

$$a_{k+1}\alpha_{k+1} + \cdots + a_n\alpha_n = 0. \tag{4}$$

   But $\alpha_{k+1}, \ldots, \alpha_n$ are a basis for $\ker(A)$, and hence are linearly independent. Thus $a_i = 0$ for $i > k$, and hence all the coefficients must be 0. □

   Let T be the linear transformation induced by multiplication by A: $T(\alpha) = A\alpha$. Then $[T]_{\mathfrak{B}}$ is similar to A. The i-th column of $[T]_{\mathfrak{B}}$ is $[T(\alpha_i)]_{\mathfrak{B}}$. By Claim 1,

$$T(\alpha_i) = \begin{cases} \alpha_i & \text{for } i \le k \\ 0 & \text{for } i > 0. \end{cases}$$

Thus, the i-th column of $[T]_{\mathfrak{B}}$ is e_i if $i \leq k$, and is 0 if $i > k$, so $[T]_{\mathfrak{B}} = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right]$.

4. A matrix is upper triangular if the entries below the diagonal are all 0 (i.e., $A_{ij} = 0$ for $i > j$). Using either the permutation formula or the expansion with respect to some row or column, show that if A is upper triangular, then $\det A = A_{11} \dots A_{nn}$, the product of the diagonal entries.

   Solution: We argue by induction on n. For $n = 1$, every $n \times n$ matrix is upper triangular, and $\det A = A_{11}$, so the result is true.

   Suppose inductively that $n > 1$ and that the determinant of every $(n-1) \times (n-1)$ upper triangular matrix is the product of its diagonal entries.

   Let A be an $n \times n$ upper triangular matrix. We apply the expansion of $\det A$ with respect to the last row:

$$\det A = \sum_{j=1}^{n} A_{nj}(-1)^{n+j} \det A(n|j).$$

   Because A is upper triangular, $A_{nj} = 0$ for $j < n$, so

$$\det A = A_{nn}(-1)^{n+n} \det A(n|n).$$

   But $A(n|n)$ is an upper triangular $(n-1) \times (n-1)$ matrix whose determinant is equal to $A_{11} \dots A_{n-1,n-1}$ by induction. The result follows.

5. Let $A, B \in F^{n \times n}$ be similar. Show that $xI - A$ amd $xI - B$ are similar in $F[x]^{n \times n}$. Deduce that $\mathrm{ch}_A(x) = \mathrm{ch}_B(x)$. (Recall that $\mathrm{ch}_A(x) = \det(xI - A)$.)

   Solution: Let $P \in F^{n \times n}$ with $P^{-1}AP = B$. Then

$$\begin{aligned} P^{-1}(xI_n - A)P &= P^{-1}xI_nP - P^{-1}AP \\ &= xI_n - B \end{aligned}$$

   because xI_n is a scalar matrix, and commutes with every element of $F[x]^{n \times n}$.

   Since similar matrices have the same determinant, the result follows.

6. Let $A \in F^{n \times n}$ with $A^2 = A$. Show that $\mathrm{ch}_A(x) = (x-1)^k x^{n-k}$ for some $k \leq n$.

Solution: By Problem 3, A is similar to $B = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right]$. Now B is upper triangular, hence so is $xI_n - B$, so $\mathrm{ch}_B(x) = \det(xI_n - B)$ is the product of the diagonal entries of $xI_n - B$ (by Problem 4). But that is precisely the stated expression. But Problem 5 shows $\mathrm{ch}_A(x) = \mathrm{ch}_B(x)$.

7. Let $A \in F^{n \times n}$ with $A^2 = 0$. Show that $\mathrm{ch}_A(x) = x^n$.
   Solution: We need an analogue of Problem 3:

Claim 3. *Let $A \in F^{n\times n}$ with $A^2 = 0$. Then A is similar to a matrix of the form*

$$B = \left[\begin{array}{c|c} 0 & * \\ \hline 0 & 0 \end{array}\right],$$

where the 0 in the upper left hand corner is the $k \times k$ 0 matrix, where $k = \dim\ker A$, the 0 in the lower right is the $(n-k)\times(n-k)$ 0 matrix, and $*$ is a $k \times (n-k)$ matrix which may contain nonzero entries.

Proof. Let $\alpha_1, \dots, \alpha_k$ be a basis for $\ker A$. Extend it to a basis $\mathfrak{B} = \alpha_1, \dots, \alpha_n$ of F^n. We shall show that if T is the linear tranformation induced by multiplication by A (i.e., $T(\alpha) = A\alpha$ for all α), then $[T]_{\mathfrak{B}}$ has the desired form.

To see this, recall that the i-th column of $[T]_{\mathfrak{B}}$ is $[T(\alpha_i)]_{\mathfrak{B}}$. For $i \le k$, $\alpha_i \in \ker A$, and hence $T(\alpha_i) = 0$. Thus, the left hand blocks of $[T]_{\mathfrak{B}}$ are 0's, as claimed.

To see that the lower right hand block of $[T]_{\mathfrak{B}}$ is 0, it suffices to show that if $i > k$, then $T(\alpha_i) \in \mathrm{Span}(\alpha_1, \dots, \alpha_k) = \ker A$. But $T(\alpha_i) = A\alpha_i$. Since $A^2 = 0$, $A \cdot A\beta = 0$ for all β, hence $A\beta \in \ker A$ for all $\beta \in F^n$. □

The matrix B given by Claim 3 is upper triangular, hence $xI_n - B$ is, also. Thus, $\mathrm{ch}_B(x)$ is the product of the diagonal entries of $xI_n - B$. Since the diagonal entries of B are all 0, the diagonal entries of $xI_n - B$ are all x, so $\mathrm{ch}_B(x) = x^n$.

But $\mathrm{ch}_B(x) = \mathrm{ch}_A(x)$ by Problem 5.