1. Let $B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. Define $T : \mathbb{Q}^{2 \times 2} \rightarrow \mathbb{Q}^{2 \times 2}$ by $T(A) = BA$. Give bases for the kernel and range of $T$.

**Solution:** Conceptually, $BA = 0$ if the range of $A$ is contained in the kernel of $B$. $B$ reduces to $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$, so the kernel of $B$ is $\text{Span}([-3, 1])$.

The range of $A$ is the span of its columns. Thus, $A$ is in $\text{ker} T$ if each of its columns is in $\text{Span}([[-3], [-1]])$, i.e., if $A = \begin{bmatrix} a(-3) & b(-3) \\ a(1) & b(1) \end{bmatrix}$ for some $a, b \in \mathbb{Q}$.

We can also derive this directly, without discussing the range of $A$ and the kernel of $B$: If $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, then

$$BA = \begin{bmatrix} x + 3z & y + 3w \\ 2(x + 3z) & 2(y + 3w) \end{bmatrix},$$

so $A \in \text{ker} T$ if and only if $x + 3z = 0$ and $y + 3w = 0$, i.e., if and only if $x = -3z$ and $y = -3w$. But then $\begin{bmatrix} x \\ z \end{bmatrix} = z \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} y \\ w \end{bmatrix} = w \cdot \begin{bmatrix} -3 \end{bmatrix}$.

Thus, the elements of $\text{ker} T$ are linear combinations of $A_1 = \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix}$. The matrices $A_1$ and $A_2$ are linearly independent, as $aA_1 + bA_2 = \begin{bmatrix} a(-3) & b(-3) \\ a(1) & b(1) \end{bmatrix}$. If this is 0, then $-3a = 0$ and $-3b = 0$, so $a = b = 0$.

Thus, $A_1$ and $A_2$ form a basis for $\text{ker} T$.

For the range of $T$, we can again use the calculation that if $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, then

$$BA = \begin{bmatrix} x + 3z & y + 3w \\ 2(x + 3z) & 2(y + 3w) \end{bmatrix}
= \begin{bmatrix} a \cdot 1 & b \cdot 1 \\ a \cdot 2 & b \cdot 2 \end{bmatrix},$$

where $a = x + 3z$ and $b = y + 3w$. Since $x, y, z, w$ are arbitrary, so are $a$ and $b$, so the range of $T$ is $\text{Span}([1, 0], [0, 1])$. As above, these two matrices are linearly independent, and hence form a basis for the range of $T$. 

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2. Let \( t_1, \ldots, t_k \in F \) be distinct. Let \( p(x) = (x - t_1) \ldots (x - t_k) \). Let \( A \in F^{n \times n} \) with \( p(A) = 0 \), i.e., \((A - t_1 I_n) \ldots (A - t_k I_n) = 0\).

Let \( P_1(x), \ldots, P_k(x) \) be the Lagrange polynomials for \( t_1, \ldots, t_k \), i.e.,

\[
P_i(x) = \frac{\prod_{j \neq i} (x - t_j)}{\prod_{j \neq i} (t_i - t_j)}.
\]

Let \( B_i = P_i(A) = \prod_{j \neq i} (A - t_j I_n) \). Show the following:

a) \( B_1 + \cdots + B_k = I_n \).
b) \( B_i B_j = 0 \) for \( i \neq j \).
c) \( B_i^2 = B_i \) for all \( i \).
d) \( t_1 B_1 + \cdots + t_k B_k = A \).

(Hint: What are \( P_1(x) + \cdots + P_k(x) \) and \( t_1 P_1(x) + \cdots + t_k P_k(x) \)?)

Solution: We first solve the hint. Recall that if \( f \) is a polynomial of degree less than \( k \), then formula (4-14) in section 4.3 gives:

\[
f = f(t_i) P_1 + \cdots + f(t_k) P_k.
\]

Apply this first to \( f = 1 \). Since \( f \) is a the constant function whose value is 1 on every point, we see that \( 1 = P_1 + \cdots + P_k \). Now \( 1(A) = I_n \), so

\[
I_n = P_1(A) + \cdots + P_k(A) = B_1 + \cdots + B_k,
\]

so a) is true.

Next, we apply the same formula to \( f = x \). Evaluating \( x \) at \( t_i \) gives \( t_i \), so \( x = t_1 P_1 + \cdots + t_k P_k \). Evaluating these polynomials at \( A \) gives

\[
A = t_1 B_1 + \cdots + t_k B_k,
\]

so d) is true.

We now prove b). Note first that \( \prod_{k \neq i} (x - t_k) \) divides \( P_i \). For the same reason, \( (x - t_i) \) divides \( P_j \) if \( i \neq j \). Thus \( (x - t_1) \ldots (x - t_k) \) divides \( P_i P_j \). Thus \( 0 = (A - t_1 I) \ldots (A - t_k I) \) divides \( P_i(A) P_j(A) = B_i B_j \), so \( B_i B_j = 0 \).

To obtain c), multiply both sides of a) on the left by \( B_i \):

\[
B_i B_1 + \cdots + B_i B_k = B_i.
\]

For \( i \neq j \), \( B_i B_j = 0 \), so the left hand side is just \( B_i^2 \), giving c).
3. Let $A \in F^{n \times n}$ with $A^2 = A$. Show that $A$ is similar to \[
\begin{bmatrix}
I_k & 0 \\
0 & 0
\end{bmatrix}
\] for some $k \leq n$.

\textit{Solution:} We first show the following:

\textbf{Claim 1.} If $\alpha$ is in the range of $A$, $A\alpha = \alpha$.

\textbf{Proof.} Since $\alpha$ is in the range of $A$, $\alpha = A\beta$ for some $\beta$. Thus $A\alpha = A \cdot A\beta = A^2\beta = A\beta = \alpha$, since $A^2 = A$. \hfill $\square$

Now let $\alpha_1, \ldots, \alpha_k$ be a basis for $\text{Range}(A)$. Since

$$\dim \text{Range}(A) + \dim \ker(A) = n,$$

the kernel of $A$ has dimension $n - k$. Let $\alpha_{k+1}, \ldots, \alpha_n$ be a basis for $\ker(A)$.

\textbf{Claim 2.} With the choices above, $\mathcal{B} = \alpha_1, \ldots, \alpha_n$ is a basis for $F^n$.

\textbf{Proof.} Since $F^n$ has dimension $n$, it suffices to show that $\alpha_1, \ldots, \alpha_n$ are linearly independent. Suppose that

$$a_1\alpha_1 + \cdots + a_n\alpha_n = 0.$$  \hfill (1)

Then multiplication by $A$ gives

$$a_1A\alpha_1 + \cdots + a_nA\alpha_n = 0.$$  \hfill (2)

For $i > k$, $\alpha_i \in \ker(A)$, so $A\alpha_i = 0$. For $i \leq k$, $A\alpha_i = \alpha_i$ by Claim 1. Thus, (2) reduces to

$$a_1\alpha_1 + \cdots + a_k\alpha_k = 0.$$  \hfill (3)

But $\alpha_1, \ldots, \alpha_k$ is a basis of $\text{Range}(A)$, and hence is linearly independent. Thus $a_i = 0$ for $i \leq k$. But then (1) reduces to

$$a_{k+1}\alpha_{k+1} + \cdots + a_n\alpha_n = 0.$$  \hfill (4)

But $\alpha_{k+1}, \ldots, \alpha_n$ are a basis for $\ker(A)$, and hence are linearly independent. Thus $a_i = 0$ for $i > k$, and hence all the coefficients must be 0. \hfill $\square$

Let $T$ be the linear transformation induced by multiplication by $A$: $T(\alpha) = A\alpha$. Then $[T]_{\mathcal{B}}$ is similar to $A$. The $i$-th column of $[T]_{\mathcal{B}}$ is $[T(\alpha_i)]_{\mathcal{B}}$. By Claim 1,

$$T(\alpha_i) = \begin{cases} 
\alpha_i & \text{for } i \leq k \\
0 & \text{for } i > k.
\end{cases}$$
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Thus, the $i$-th column of $[T]_B$ is $e_i$ if $i \leq k$, and is 0 if $i > k$, so

$$[T]_B = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}.$$

4. A matrix is upper triangular if the entries below the diagonal are all 0 (i.e., $A_{ij} = 0$ for $i > j$). Using either the permutation formula or the expansion with respect to some row or column, show that if $A$ is upper triangular, then $\det A = A_{11} \ldots A_{nn}$, the product of the diagonal entries.

Solution: We argue by induction on $n$. For $n = 1$, every $n \times n$ matrix is upper triangular, and $\det A = A_{11}$, so the result is true.

Suppose inductively that $n > 1$ and that the determinant of every $(n-1) \times (n-1)$ upper triangular matrix is the product of its diagonal entries.

Let $A$ be an $n \times n$ upper triangular matrix. We apply the expansion of $\det A$ with respect to the last row:

$$\det A = \sum_{j=1}^{n} A_{nj}(-1)^{n+j} \det A(n|j).$$

Because $A$ is upper triangular, $A_{nj} = 0$ for $j < n$, so

$$\det A = A_{nn}(-1)^{n+n} \det A(n|n).$$

But $A(n|n)$ is an upper triangular $(n-1) \times (n-1)$ matrix whose determinant is equal to $A_{11} \ldots A_{n-1,n-1}$ by induction. The result follows.

5. Let $A, B \in F^{n \times n}$ be similar. Show that $xI - A$ and $xI - B$ are similar in $F[x]^{n \times n}$. Deduce that $\text{ch}_A(x) = \text{ch}_B(x)$. (Recall that $\text{ch}_A(x) = \det(xI - A)$.)

Solution: Let $P \in F^{n \times n}$ with $P^{-1}AP = B$. Then

$$P^{-1}(xI_n - A)P = P^{-1}xI_nP - P^{-1}AP = xI_n - B$$

because $xI_n$ is a scalar matrix, and commutes with every element of $F[x]^{n \times n}$. Since similar matrices have the same determinant, the result follows.

6. Let $A \in F^{n \times n}$ with $A^2 = A$. Show that $\text{ch}_A(x) = (x-1)^k x^{n-k}$ for some $k \leq n$.
Solution: By Problem 3, $A$ is similar to $B = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$. Now $B$ is upper triangular, hence so is $xI_n - B$, so $\text{ch}_B(x) = \det(xI_n - B)$ is the product of the diagonal entries of $xI_n - B$ (by Problem 4). But that is precisely the stated expression. But Problem 5 shows $\text{ch}_A(x) = \text{ch}_B(x)$.

7. Let $A \in F^{n \times n}$ with $A^2 = 0$. Show that $\text{ch}_A(x) = x^n$.

Solution: We need an analogue of Problem 3:

Claim 3. Let $A \in F^{n \times n}$ with $A^2 = 0$. Then $A$ is similar to a matrix of the form

$$B = \begin{bmatrix} 0 & \ast \\ 0 & 0 \end{bmatrix},$$

where the $0$ in the upper left hand corner is the $k \times k$ $0$ matrix, where $k = \dim \ker A$, the $0$ in the lower right is the $(n-k) \times (n-k)$ $0$ matrix, and $\ast$ is a $k \times (n-k)$ matrix which may contain nonzero entries.

Proof. Let $\alpha_1, \ldots, \alpha_k$ be a basis for $\ker A$. Extend it to a basis $B = \alpha_1, \ldots, \alpha_n$ of $F^n$. We shall show that if $T$ is the linear tranformation induced by multiplication by $A$ (i.e., $T(\alpha) = A\alpha$ for all $\alpha$), then $[T]_B$ has the desired form.

To see this, recall that the $i$-th column of $[T]_B$ is $[T(\alpha_i)]_B$. For $i \leq k$, $\alpha_i \in \ker A$, and hence $T(\alpha_i) = 0$. Thus, the left hand blocks of $[T]_B$ are $0$’s, as claimed.

To see that the lower right hand block of $[T]_B$ is $0$, it suffices to show that if $i > k$, then $T(\alpha_i) \in \text{Span}(\alpha_1, \ldots, \alpha_k) = \ker A$. But $T(\alpha_i) = A\alpha_i$. Since $A^2 = 0$, $A \cdot A\beta = 0$ for all $\beta$, hence $A\beta \in \ker A$ for all $\beta \in F^n$.

The matrix $B$ given by Claim 3 is upper triangular, hence $xI_n - B$ is, also. Thus, $\text{ch}_B(x)$ is the product of the diagonal entries of $xI_n - B$. Since the diagonal entries of $B$ are all $0$, the diagonal entries of $xI_n - B$ are all $x$, so $\text{ch}_B(x) = x^n$.

But $\text{ch}_B(x) = \text{ch}_A(x)$ by Problem 5.