Math 424-524 **Midterm** **Fall '97**

1. Let $B = \left[\begin{smallmatrix} 1 & 3 \\ 2 & 6 \end{smallmatrix}\right]$. Define $T : \mathbf{Q}^{2\times 2} \to \mathbf{Q}^{2\times 2}$ by $T(A) = BA$. Give bases for the kernel and range of T.

2. Let $t_1, \ldots, t_k \in F$ be distinct. Let $p(x) = (x - t_1) \ldots (x - t_k)$. Let $A \in F^{n\times n}$ with $p(A) = 0$, i.e., $(A - t_1 I_n) \ldots (A - t_k I_n) = 0$.

   Let $P_1(x), \ldots, P_k(x)$ be the Lagrange polynomials for $t_1, \ldots, t_k$, i.e.,
$$P_i(x) = \frac{\prod_{j\neq i}(x - t_j)}{\prod_{j\neq i}(t_i - t_j)}.$$
   Let $B_i = P_i(A) = \frac{1}{\prod_{j\neq i}(t_i - t_j)} \prod_{j\neq i}(A - t_j I_n)$. Show the following:

   a) $B_1 + \cdots + B_k = I_n$.
   b) $B_i B_j = 0$ for $i \neq j$.
   c) $B_i^2 = B_i$ for all i.
   d) $t_1 B_1 + \cdots + t_k B_k = A$.

   (Hint: What are $P_1(x) + \cdots + P_k(x)$ and $t_1 P_1(x) + \cdots + t_k P_k(x)$?)

3. Let $A \in F^{n\times n}$ with $A^2 = A$. Show that A is similar to $\left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right]$ for some $k \leq n$.

4. A matrix is upper triangular if the entries below the diagonal are all 0 (i.e., $A_{ij} = 0$ for $i > j$). Using either the permutation formula or the expansion with respect to some row or column, show that if A is upper triangular, then $\det A = A_{11} \ldots A_{nn}$, the product of the diagonal entries.

5. Let $A, B \in F^{n\times n}$ be similar. Show that $xI - A$ amd $xI - B$ are similar in $F[x]^{n\times n}$. Deduce that $\mathrm{ch}_A(x) = \mathrm{ch}_B(x)$. (Recall that $\mathrm{ch}_A(x) = \det(xI - A)$.)

6. Let $A \in F^{n\times n}$ with $A^2 = A$. Show that $\mathrm{ch}_A(x) = (x - 1)^k x^{n-k}$ for some $k \leq n$.

7. Let $A \in F^{n\times n}$ with $A^2 = 0$. Show that $\mathrm{ch}_A(x) = x^n$.