1. Let \( B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \). Define \( T : \mathbb{Q}^{2 \times 2} \rightarrow \mathbb{Q}^{2 \times 2} \) by \( T(A) = BA \). Give bases for the kernel and range of \( T \).

2. Let \( t_1, \ldots, t_k \in F \) be distinct. Let \( p(x) = (x - t_1) \cdots (x - t_k) \). Let \( A \in F^{n \times n} \) with \( p(A) = 0 \), i.e., \((A - t_1 I_n) \cdots (A - t_k I_n) = 0\). Let \( P_1(x), \ldots, P_k(x) \) be the Lagrange polynomials for \( t_1, \ldots, t_k \), i.e.,

\[
P_i(x) = \prod_{j \neq i} \frac{(x - t_j)}{(t_i - t_j)}.
\]

Let \( B_i = P_i(A) = \frac{1}{\prod_{j \neq i} (t_i - t_j)} \prod_{j \neq i} (A - t_j I_n) \). Show the following:

a) \( B_1 + \cdots + B_k = I_n \).

b) \( B_i B_j = 0 \) for \( i \neq j \).

c) \( B_i^2 = B_i \) for all \( i \).

d) \( t_1 B_1 + \cdots + t_k B_k = A \).

(Hint: What are \( P_1(x) + \cdots + P_k(x) \) and \( t_1 P_1(x) + \cdots + t_k P_k(x) \)?)

3. Let \( A \in F^{n \times n} \) with \( A^2 = A \). Show that \( A \) is similar to

\[
\begin{bmatrix}
I_k & 0 \\
0 & 0
\end{bmatrix}
\]

for some \( k \leq n \).

4. A matrix is upper triangular if the entries below the diagonal are all 0 (i.e., \( A_{ij} = 0 \) for \( i > j \)). Using either the permutation formula or the expansion with respect to some row or column, show that if \( A \) is upper triangular, then \( \det(A) = A_{11} \cdots A_{nn} \), the product of the diagonal entries.

5. Let \( A, B \in F^{n \times n} \) be similar. Show that \( xI - A \) and \( xI - B \) are similar in \( F[x]^{n \times n} \). Deduce that \( \text{ch}_A(x) = \text{ch}_B(x) \). (Recall that \( \text{ch}_A(x) = \det(xI - A) \).)

6. Let \( A \in F^{n \times n} \) with \( A^2 = A \). Show that \( \text{ch}_A(x) = (x - 1)^k x^{n-k} \) for some \( k \leq n \).

7. Let \( A \in F^{n \times n} \) with \( A^2 = 0 \). Show that \( \text{ch}_A(x) = x^n \).