1. We consider $n \times n$ matrices $A$ for which $\text{ch}_A(x) = (x - 5)^n$.

a) What is the smallest value of $n$ for which the Jordan canonical form of $A$ is not determined by knowing $\text{min}_A(x)$? Give an example of two different Jordan canonical forms for this $n$ that have the same minimal polynomial.

b) What is the smallest value of $n$ for which the Jordan canonical form of $A$ is not determined by knowing the dimension of the eigenspace of $A; 5$? Give an example of two different Jordan canonical forms for this $n$ whose eigenspaces have the same dimension.

c) What is the smallest value of $n$ for which the Jordan canonical form of $A$ is not determined by knowing both $\text{min}_A(x)$ and $\text{dim} \ N(A - 5I)$? Give an example of two different Jordan canonical forms for this $n$ that have the same minimal polynomial and whose eigenspaces have the same dimension.

d) What is the smallest value of $n$ for which the Jordan canonical form of $A$ is not determined by knowing $\text{min}_A(x)$, $\text{dim} \ N(A - 5I)$ and $\text{dim} \ N(A - 5I)^2$? Give an example of two different Jordan canonical forms for this $n$ for which these three invariants agree.

2. Let

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14 & -12 & -1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ -16 & 1 & 2 & -1 & 1 & 3 & 0 & 0 & 0 & 0 \\ -14 & 1 & 3 & -1 & 1 & 0 & 3 & 0 & 0 & 0 \\ -9 & -10 & 2 & -3 & 2 & -1 & 1 & 3 & 0 & 0 \\ 22 & -13 & -7 & 3 & 2 & -2 & 0 & 1 & 3 & 0 \\ -32 & -5 & 16 & -8 & 0 & 0 & 2 & -1 & 0 & 3 \end{bmatrix}. $$

a) What is the characteristic polynomial of $A$?
b) For each eigenvalue $c$, find $\dim \ker (A - cI)^k$ for each $k$ less than or equal to the multiplicity of $c$ in $\text{ch}_A(x)$. (To show your work, display the reduced row echelon form for each $(A - cI)^k$ that you need for your argument. You may use maple to reduce the matrices.)

c) What is the minimal polynomial of $A$?

d) What is the Jordan canonical form of $A$? (Write it as a formal block sum of formal Jordan blocks.)

e) If you know the minimal polynomial, for how many values of $k$, beginning with $k = 1$ and incrementing by ones, do you need to calculate $\dim \ker (A - 3I)^k$ in order to determine the Jordan canonical form?

f) If you don’t know the minimal polynomial, for how many values of $k$, beginning with $k = 1$ and incrementing by ones, do you need to calculate $\dim \ker (A - 3I)^k$ in order to determine the Jordan canonical form?

3. Find all irreducible-power rational polynomials $f$ such that the matrix $A = C(f)$ satisfies $A^8 = I$. (Hint: $\text{min}_A(x)$ must divide $x^8 - 1$.) Use this to give the prime-power decomposition “rational form” for all rational $n \times n$ matrices $A$ with $A^8 = I$, where $n$ is:

a) 4,

b) 8.