Math 367 Final Exam Solutions Summer '99

1. What is the probability of getting a full house in 7 card poker (i.e., three cards in one denomination and at least two cards in at least one other denomination, and no more than three of any denomination)?

   SOLUTION: The possible distributions of cards in denominations are 3,3,1, 3,2,2, 3,2,1,1. The probability is

$$\frac{\binom{13}{2}\binom{4}{3}^2\binom{11}{1}\binom{4}{1} + \binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{2}^2 + \binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}\binom{11}{2}\binom{4}{1}}{\binom{52}{7}}$$

2. An urn contains 12 balls each of the following colors: Red, Blue, Green, White, Yellow.

   a) Draw 7 times without replacement from the urn. Find the
      (i) expected value
      (ii) variance
      for the number of red balls drawn.

   SOLUTION: Here, $X = X_1 + \cdots + X_7$, where

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th ball is red} \\ 0 & \text{otherwise.} \end{cases}$$

   We have

$$E(X_i) = P(i\text{-th ball is red}) = \frac{12}{60} = \frac{1}{5},$$

   and

$$\begin{aligned} E(X_iX_j) &= P(i\text{-th and } j\text{-th balls red}) \\ &= P(j\text{-th red} \mid i\text{-th red})P(i\text{-th red}) \\ &= \frac{11}{59} \cdot \frac{12}{60} = \frac{11}{295}. \end{aligned}$$

   Thus, $E(X) = E(X_1) + \cdots + E(X_7) = 7 \cdot \frac{1}{5} = \frac{7}{5}$, and

$$\begin{aligned} \text{Var}(X) &= E(X^2) - E(X)^2 \\ &= E(X_1) + \cdots + E(X_7) + \sum_{i \neq j} E(X_iX_j) - E(X)^2 \\ &= 7 \cdot \frac{1}{5} + 7 \cdot 6 \cdot \frac{11}{295} - \left(\frac{7}{5}\right)^2 \end{aligned}$$

b) Draw 7 times without replacement from the urn. Find the
(i) expected value
(ii) variance
for the number of colors drawn.

SOLUTION: Here, $X = 5 - Y$, where Y is the number of colors not drawn, and $Y = Y_1 + \cdots + Y_5$, with

$$Y_i = \begin{cases} 1 & \text{if the } i\text{-th color is not drawn} \\ 0 & \text{otherwise.} \end{cases}$$

So

$$E(Y_i) = P(i\text{-th color not drawn}) = \frac{\binom{48}{7}}{\binom{60}{7}},$$

as there are 48 balls not of the i-th color, and

$$E(Y_iY_j) = P(i\text{-th and } j\text{-th colors not drawn}) = \frac{\binom{36}{7}}{\binom{60}{7}},$$

as there are 36 balls whose color is different from the i-th and j-th colors. Thus,

$$E(Y) = E(Y_1) + \cdots + E(Y_5) = 5 \cdot \frac{\binom{48}{7}}{\binom{60}{7}},$$

and

$$\begin{aligned} \mathrm{Var}(Y) &= E(Y^2) - E(Y)^2 \\ &= E(Y_1) + \cdots + E(Y_5) + \sum_{i \neq j} E(Y_iY_j) - E(Y)^2 \\ &= 5 \cdot \frac{\binom{48}{7}}{\binom{60}{7}} + 5 \cdot 4 \cdot \frac{\binom{36}{7}}{\binom{60}{7}} - \left(5 \cdot \frac{\binom{48}{7}}{\binom{60}{7}}\right)^2. \end{aligned}$$

Since $X = 5 - Y$,

$$E(X) = 5 - E(Y) = 5 \cdot \left(1 - \frac{\binom{48}{7}}{\binom{60}{7}}\right)$$

and $\mathrm{Var}(X) = \mathrm{Var}(Y)$.

c) Draw 7 times with replacement from the urn. Find the
   (i) expected value
   (ii) variance
   for the number of colors drawn.

Solution: Again, $X = 5 - Y$, where Y is the number of colors not drawn, and $Y = Y_1 + \cdots + Y_5$, with

$$Y_i = \begin{cases} 1 & \text{if the } i\text{-th color is not drawn} \\ 0 & \text{otherwise.} \end{cases}$$

This time, the probability the i-th color is not drawn on a given draw is $\frac{48}{60} = \frac{4}{5}$ (as was also the case for part b), and since the drawings are with replacement, the probability the i-th color is not drawn at all is $(\frac{4}{5})^7$. Thus, $E(Y_i) = (\frac{4}{5})^7$, and

$$E(Y) = E(Y_1) + \cdots + E(Y_5) = 5\left(\frac{4}{5}\right)^7.$$

The probability that neither the i-th nor j-th colors are drawn on a given draw is $\frac{36}{60} = \frac{3}{5}$, so $E(Y_iY_j) = (\frac{3}{5})^7$. Thus,

$$\begin{aligned} \mathrm{Var}(Y) &= E(Y^2) - E(Y)^2 \\ &= E(Y_1) + \cdots + E(Y_5) + \sum_{i \neq j} E(Y_iY_j) - E(Y)^2 \\ &= 5\left(\frac{4}{5}\right)^7 + 5 \cdot 4\left(\frac{3}{5}\right)^7 - \left(5\left(\frac{4}{5}\right)^7\right)^2. \end{aligned}$$

Again, $\mathrm{Var}(X) = \mathrm{Var}(Y)$, and

$$E(X) = 5 - E(Y) = 5\left(1 - \left(\frac{4}{5}\right)^7\right).$$

3. A pile contains 400 normal coins and one trick coin that lands heads 75% of the time. You pick a coin from the pile at random and toss it five times. It lands heads four times out of five.

   a) What is the probability your coin is the trick coin?

   SOLUTION: With the trick coin, the probability of getting 4 heads out of 5 is $\binom{5}{4}(\frac{3}{4})^4\frac{1}{4}$. An ordinary coin has a probability of $\binom{5}{4}(\frac{1}{2})^4\frac{1}{2}$ of 4 heads in 5 tosses. Thus,

$$\begin{aligned} P(\text{trick coin} \,|\, \text{4H out of 5}) &= \frac{P(\text{trick coin and 4H out of 5})}{P(\text{4H out of 5})} \\ &= \frac{\frac{1}{401} \cdot \binom{5}{4} \left(\frac{3}{4}\right)^4 \frac{1}{4}}{\frac{1}{401} \cdot \binom{5}{4} \left(\frac{3}{4}\right)^4 \frac{1}{4} + \frac{400}{401} \cdot \binom{5}{4} \left(\frac{1}{2}\right)^4 \frac{1}{2}} \\ &= x \approx .63\% \end{aligned}$$

   We shall use this number x in part b).

   b) If you toss your coin again, what is the probability it falls heads?

   SOLUTION: In part a) we calculated x, the probability that you picked the trick coin, given that it landed heads four out of five tries. Note that $(1 - x)$ is the probability that it's *not* the trick coin, given that it landed heads four out of five tries.

   Since the trick coin has a probability of $\frac{3}{4}$ of landing heads, and a normal coin has a probability of $\frac{1}{2}$ of landing heads, we have

$$P(\text{6-th toss H} \,|\, \text{4H out of 5}) = x \cdot \frac{3}{4} + (1 - x) \cdot \frac{1}{2} \approx 50.16\%$$

4. A marksman hits 90% of his shots. He shoots until he gets 20 hits.
   a) What is the probability he does this within 22 shots?

   SOLUTION: We're performing Bernoulli trials with $p = .9$. Recall that the probability that the k-th success occurrs on the n-th trial is $\binom{n-1}{k-1} p^k q^{n-k}$. Thus,

$$\begin{aligned} P(20 \text{ hits in } \leq 22) &= P(20\text{th hit on }20\text{th}) + P(20\text{th hit on }21\text{st}) \\ &\quad + P(20\text{th hit on }22\text{nd}) \\ &= \binom{19}{19}(.9)^{20}(.1)^0 + \binom{20}{19}(.9)^{20}(.1)^1 \\ &\quad + \binom{21}{19}(.9)^{20}(.1)^2. \end{aligned}$$

   b) What is the expected number of shots it will take?

   SOLUTION: The expected number of trials needed for r successes is $\frac{r}{p}$, which comes to $\frac{20}{.9} = 22\frac{2}{9}$, in this case.

5. A committee consists of 6 men and 8 women. A subcommittee of 4 is chosen. What is the probability it contains at least two women if it contains at most three men?

   SOLUTION: The committee has at most three men if and only if it contains at least 1 woman. Thus, we wish to calculate

$$\begin{aligned} P(\geq 2W \mid \geq 1W) &= \frac{P(\geq 2W)}{P(\geq 1W)} \\ &= \frac{[\binom{8}{2}\binom{6}{2} + \binom{8}{3}\binom{6}{1} + \binom{8}{4}\binom{6}{0}]/\binom{14}{4}}{[\binom{8}{1}\binom{6}{3} + \binom{8}{2}\binom{6}{2} + \binom{8}{3}\binom{6}{1} + \binom{8}{4}\binom{6}{0}]/\binom{14}{4}} \end{aligned}$$

6. Peter has \$75 and Paul has \$50. They play a game in which they bet \$1 on each play, even money, and Peter has a 40% chance of winning each play.

   a) What is the probability Peter gets \$25 ahead before he loses all his money?

      SOLUTION: According to the rules of play, we can assume that Paul has \$25 and will stop when he loses it. Thus, the total amount of money in the game is $t = 100$, and Peter has $s = 75$. We have $r = \frac{.60}{.40} = 1.5$, and the probability Peter wins is

      $$p^* = \frac{1 - (1.5)^{75}}{1 - (1.5)^{100}}.$$

   b) How long should it take for one or the other of these to happen?

   SOLUTION: The duration of play is

   $$\frac{p^* t - s}{p - q} = \frac{100p^* - 75}{-.2},$$

   where p^* is the answer to part a).

7. Consider the transition matrix

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

a) Find R_i for each state $i = 1, \ldots, 8$.

Solution:

$$\begin{aligned} R_1 &= \{E_1, E_4, E_7\} \\ R_2 &= \{E_2, E_5, E_8\} \\ R_3 &= \{E_2, E_4, E_6, E_5, E_8, E_7, E_3, E_1\} \\ R_4 &= \{E_4, E_7, E_1\} \\ R_5 &= \{E_2, E_8, E_5\} \\ R_6 &= \{E_3, E_5, E_6, E_2, E_4, E_8, E_7, E_1\} \\ R_7 &= \{E_1, E_7, E_4\} \\ R_8 &= \{E_5, E_2, E_8\} \end{aligned}$$

b) Classify the recurrent and the transient states.

Solution: E_3 is transient, because $E_1 \in R_3$ but $E_3 \notin R_1$, i.e., you can get to E_1 from E_3, but you can't get to E_3 from E_1.

Similarly, E_6 is transient, because $E_1 \in R_6$ but $E_6 \notin R_1$.

All other states are recurrent.

c) Compute h_{33}.

SOLUTION: We use the formula $h_{ij} = p_{ij} + \sum_{k \neq j} p_{ik} h_{kj}$. Here, we get

$$\begin{aligned} h_{33} &= \frac{1}{4}h_{23} + \frac{1}{4}h_{43} + \frac{1}{2}h_{63} \\ &= \frac{1}{2}h_{63}, \end{aligned}$$

as $E_3 \notin R_2$ and $E_3 \notin R_4$, and hence $h_{23} = h_{43} = 0$. The same formula gives

$$\begin{aligned} h_{63} &= \frac{1}{3} + \frac{1}{3}h_{53} + \frac{1}{3}h_{63} \\ &= \frac{1}{3} + \frac{1}{3}h_{63}, \end{aligned}$$

as $E_3 \notin R_5$, so that $h_{53} = 0$. Solving this last equation gives $h_{63} = \frac{1}{2}$. Substituting this in the first equation gives $h_{33} = \frac{1}{4}$.

d) For each transient state, compute the expected number of steps needed to reach a recurrent state.

SOLUTION: We combine all the recurrent states into a single, absorbing state E_0. Our matrix then becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Here, the first row is for E_0, the second for E_3, and the third for E_6. We use the formula $r_{ij} = 1 + \sum_{k \neq j} p_{ik} r_{kj}$. This gives

$$\begin{aligned} r_{30} &= 1 + \frac{1}{2}r_{60} \\ r_{60} &= 1 + \frac{1}{3}r_{30} + \frac{1}{3}r_{60} \end{aligned}$$

Substituting the first equation into the second, we get

$$\begin{aligned} r_{60} &= 1 + \frac{1}{3}(1 + \frac{1}{2}r_{60}) + \frac{1}{3}r_{60} \\ &= \frac{4}{3} + \frac{1}{2}r_{60}. \end{aligned}$$

Solving this gives $r_{60} = \frac{8}{3}$, hence $r_{30} = \frac{7}{3}$. Thus, the expected number of steps from E_6 to a recurrent state is $\frac{8}{3}$, and from E_3 to a recurrent state is $\frac{7}{3}$.