1. What is the probability of getting a full house in 7 card poker
(i.e., three cards in one denomination and at least two cards in
at least one other denomination, and no more than three of any
denomination)?

SOLUTION: The possible distributions of cards in denominations
are 3,3,1, 3,2,2, 3,2,1,1. The probability is
\[
\left(\binom{13}{3}\right)^2 \binom{11}{4}\left(\binom{4}{3}\right)\left(\binom{1}{1}\right) + \left(\binom{13}{4}\right)\left(\binom{12}{2}\right)^2 \left(\binom{1}{1}\right)^2 + \left(\binom{13}{4}\right)\left(\binom{12}{3}\right)\left(\binom{11}{2}\right)\left(\binom{4}{3}\right)
\]
\[
\left(\binom{13}{7}\right)
\]

2. An urn contains 12 balls each of the following colors: Red, Blue,
Green, White, Yellow.
a) Draw 7 times without replacement from the urn. Find the
(i) expected value
(ii) variance
for the number of red balls drawn.

SOLUTION: Here, \(X = X_1 + \cdots + X_7\), where
\[
X_i = \begin{cases} 
1 & \text{if the } i\text{-th ball is red} \\
0 & \text{otherwise.}
\end{cases}
\]
We have
\[
E(X_i) = P(\text{i-th ball is red}) = \frac{12}{60} = \frac{1}{5},
\]
and
\[
E(X_iX_j) = P(\text{i-th and } j\text{-th balls red})
= P(\text{j-th red } \mid \text{i-th red})P(\text{i-th red})
= \frac{11}{59} \cdot \frac{12}{60} = \frac{11}{295}.
\]
Thus, \(E(X) = E(X_1) + \cdots + E(X_7) = 7 \cdot \frac{1}{5} = \frac{7}{5}\), and

Var(X) = E(X^2) − E(X)^2

\begin{align*}
&= E(X_1) + \cdots + E(X_7) + \sum_{i \neq j} E(X_iX_j) - E(X)^2 \\
&= 7 \cdot \frac{1}{5} + 7 \cdot 6 \cdot \frac{11}{295} - \left(\frac{7}{5}\right)^2
\end{align*}
Final Exam Solutions

b) Draw 7 times without replacement from the urn. Find the
(i) expected value
(ii) variance
for the number of colors drawn.

SOLUTION: Here, \( X = 5 - Y \), where \( Y \) is the number of colors not drawn, and \( Y = Y_1 + \cdots + Y_5 \), with

\[
Y_i = \begin{cases} 
1 & \text{if the } i\text{-th color is not drawn} \\
0 & \text{otherwise.}
\end{cases}
\]

So

\[
E(Y_i) = P(\text{i-th color not drawn}) = \frac{\binom{48}{7}}{\binom{60}{7}},
\]
as there are 48 balls not of the \( i \)-th color, and

\[
E(Y_i Y_j) = P(\text{i-th and } j\text{-th colors not drawn}) = \frac{\binom{36}{7}}{\binom{60}{7}},
\]
as there are 36 balls whose color is different from the \( i \)-th and \( j \)-th colors. Thus,

\[
E(Y) = E(Y_1) + \cdots + E(Y_5) = 5 \cdot \frac{\binom{48}{7}}{\binom{60}{7}},
\]
and

\[
\text{Var}(Y) = E(Y^2) - E(Y)^2
\]

\[
= E(Y_1) + \cdots + E(Y_5) + \sum_{i \neq j} E(Y_i Y_j) - E(Y)^2
\]

\[
= 5 \cdot \frac{\binom{48}{7}}{\binom{60}{7}} + 5 \cdot 4 \cdot \frac{\binom{36}{7}}{\binom{60}{7}} - \left( 5 \cdot \frac{\binom{48}{7}}{\binom{60}{7}} \right)^2.
\]

Since \( X = 5 - Y \),

\[
E(X) = 5 - E(Y) = 5 \cdot \left( 1 - \frac{\binom{48}{7}}{\binom{60}{7}} \right)
\]
and \( \text{Var}(X) = \text{Var}(Y) \).
c) Draw 7 times with replacement from the urn. Find the
(i) expected value
(ii) variance
for the number of colors drawn.

SOLUTION: Again, $X = 5 - Y$, where $Y$ is the number of colors not drawn, and $Y = Y_1 + \cdots + Y_5$, with

$$Y_i = \begin{cases} 1 & \text{if the } i\text{-th color is not drawn} \\ 0 & \text{otherwise.} \end{cases}$$

This time, the probability the $i$-th color is not drawn on a given draw is $\frac{48}{60} = \frac{4}{5}$ (as was also the case for part b), and since the drawings are with replacement, the probability the $i$-th color is not drawn at all is $\left(\frac{4}{5}\right)^7$. Thus, $E(Y_i) = \left(\frac{4}{5}\right)^7$, and

$$E(Y) = E(Y_1) + \cdots + E(Y_5) = 5 \left(\frac{4}{5}\right)^7.$$

The probability that neither the $i$-th nor $j$-th colors are drawn on a given draw is $\frac{36}{60} = \frac{3}{5}$, so $E(Y_iY_j) = \left(\frac{3}{5}\right)^7$. Thus,

$$\text{Var}(Y) = E(Y^2) - E(Y)^2
\begin{align*}
&= E(Y_1) + \cdots + E(Y_5) + \sum_{i \neq j} E(Y_iY_j) - E(Y)^2 \\
&= 5 \left(\frac{4}{5}\right)^7 + 5 \cdot 4 \left(\frac{3}{5}\right)^7 - 5 \left(\frac{4}{5}\right)^7. \\
\end{align*}$$

Again, $\text{Var}(X) = \text{Var}(Y)$, and

$$E(X) = 5 - E(Y) = 5 \left(1 - \left(\frac{4}{5}\right)^7\right).$$
Final Exam Solutions

3. A pile contains 400 normal coins and one trick coin that lands heads 75% of the time. You pick a coin from the pile at random and toss it five times. It lands heads four times out of five.

a) What is the probability your coin is the trick coin?

**Solution:** With the trick coin, the probability of getting 4 heads out of 5 is \((\binom{5}{4})(\frac{3}{4})^4\frac{1}{4}\). An ordinary coin has a probability of \((\binom{5}{4})(\frac{1}{2})^4\frac{1}{2}\) of 4 heads in 5 tosses. Thus,

\[
P(\text{trick coin} \mid 4H \text{ out of 5}) = \frac{P(\text{trick coin and 4H out of 5})}{P(4H \text{ out of 5})}
\]

\[
= \frac{\frac{1}{401} \cdot \left(\binom{5}{4}\right)\left(\frac{3}{4}\right)^4\frac{1}{4}}{\frac{1}{401} \cdot \left(\binom{5}{4}\right)\left(\frac{3}{4}\right)^4\frac{1}{4} + \frac{400}{401} \cdot \left(\binom{5}{4}\right)\left(\frac{1}{2}\right)^4\frac{1}{2}}
\]

\[
= x \approx 0.63\%
\]

We shall use this number \( x \) in part b).

b) If you toss your coin again, what is the probability it falls heads?

**Solution:** In part a) we calculated \( x \), the probability that you picked the trick coin, given that it landed heads four out of five tries. Note that \((1 - x)\) is the probability that it’s not the trick coin, given that it landed heads four out of five tries.

Since the trick coin has a probability of \(\frac{3}{4}\) of landing heads, and a normal coin has a probability of \(\frac{1}{2}\) of landing heads, we have

\[
P(\text{6-th toss H} \mid 4H \text{ out of 5}) = x \cdot \frac{3}{4} + (1 - x) \cdot \frac{1}{2} \approx 50.16\%
\]
4. A marksman hits 90% of his shots. He shoots until he gets 20 hits.
   a) What is the probability he does this within 22 shots?

   **Solution:** We’re performing Bernoulli trials with \( p = 0.9 \). Recall that the probability that the \( k \)-th success occurs on the \( n \)-th trial is \( \binom{n-1}{k-1} p^k q^{n-k} \). Thus,
   
   \[
P(20 \text{ hits in } \leq 22) = P(20\text{th hit on } 20\text{th}) + P(20\text{th hit on } 21\text{st}) + P(20\text{th hit on } 22\text{nd})
   \]
   
   \[
   = \binom{19}{19} (0.9)^{20}(0.1)^0 + \binom{20}{19} (0.9)^{20}(0.1)^1
   \]
   
   \[
   + \binom{21}{19} (0.9)^{20}(0.1)^2.
   \]

   b) What is the expected number of shots it will take?

   **Solution:** The expected number of trials needed for \( r \) successes is \( \frac{r}{p} \), which comes to \( \frac{20}{0.9} = \frac{220}{9} \), in this case.

5. A committee consists of 6 men and 8 women. A subcommittee of 4 is chosen. What is the probability it contains at least two women if it contains at most three men?

   **Solution:** The committee has at most three men if and only if it contains at least 1 woman. Thus, we wish to calculate
   
   \[
P(\geq 2W \mid \geq 1W) = \frac{P(\geq 2W)}{P(\geq 1W)}
   \]
   
   \[
   = \frac{\binom{8}{0}\binom{6}{2} + \binom{8}{1}\binom{6}{1} + \binom{8}{2}\binom{6}{0} \binom{14}{4}}{\binom{8}{1}\binom{6}{3} + \binom{8}{2}\binom{6}{2} + \binom{8}{3}\binom{6}{1} + \binom{8}{4}\binom{6}{0} \binom{14}{4}}.
   \]
6. Peter has $75 and Paul has $50. They play a game in which they bet $1 on each play, even money, and Peter has a 40% chance of winning each play.

a) What is the probability Peter gets $25 ahead before he loses all his money?

SOLUTION: According to the rules of play, we can assume that Paul has $25 and will stop when he loses it. Thus, the total amount of money in the game is $t = 100$, and Peter has $s = 75$. We have $r = \frac{40}{60} = 1.5$, and the probability Peter wins is

$$ p^* = \frac{1 - (1.5)^{75}}{1 - (1.5)^{100}}. $$

b) How long should it take for one or the other of these to happen?

SOLUTION: The duration of play is

$$ \frac{p^*t - s}{p - q} = \frac{100p^* - 75}{-2}, $$

where $p^*$ is the answer to part a).
7. Consider the transition matrix
\[
\begin{pmatrix}
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

a) Find \( R_i \) for each state \( i = 1, \ldots, 8 \).

**Solution:**

\[
\begin{align*}
R_1 & = \{E_1, E_4, E_7\} \\
R_2 & = \{E_2, E_5, E_8\} \\
R_3 & = \{E_2, E_4, E_6, E_5, E_8, E_7, E_3, E_1\} \\
R_4 & = \{E_4, E_7, E_1\} \\
R_5 & = \{E_2, E_8, E_5\} \\
R_6 & = \{E_3, E_5, E_6, E_2, E_4, E_8, E_7, E_1\} \\
R_7 & = \{E_1, E_7, E_4\} \\
R_8 & = \{E_5, E_2, E_8\}
\end{align*}
\]

b) Classify the recurrent and the transient states.

**Solution:** \( E_3 \) is transient, because \( E_1 \in R_3 \) but \( E_3 \notin R_1 \), i.e., you can get to \( E_1 \) from \( E_3 \), but you can't get to \( E_3 \) from \( E_1 \).

Similarly, \( E_6 \) is transient, because \( E_1 \in R_6 \) but \( E_6 \notin R_1 \).

All other states are recurrent.
c) Compute $h_{33}$.

SOLUTION: We use the formula $h_{ij} = p_{ij} + \sum_{k \neq j} p_{ik} h_{kj}$. Here, we get

$$h_{33} = \frac{1}{4} h_{23} + \frac{1}{4} h_{43} + \frac{1}{2} h_{63} - \frac{1}{2} h_{63},$$

as $E_3 \notin R_2$ and $E_3 \notin R_4$, and hence $h_{23} = h_{43} = 0$. The same formula gives

$$h_{63} = \frac{1}{3} h_{53} + \frac{1}{3} h_{63} = \frac{1}{3} + \frac{1}{3} h_{63},$$

as $E_3 \notin R_5$, so that $h_{53} = 0$. Solving this last equation gives $h_{63} = \frac{1}{2}$. Substituting this in the first equation gives $h_{33} = \frac{1}{4}$.

d) For each transient state, compute the expected number of steps needed to reach a recurrent state.

SOLUTION: We combine all the recurrent states into a single, absorbing state $E_0$. Our matrix then becomes

$$\begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}$$

Here, the first row is for $E_0$, the second for $E_3$, and the third for $E_6$. We use the formula $r_{ij} = 1 + \sum_{k \neq j} p_{ik} r_{kj}$. This gives

$$r_{30} = 1 + \frac{1}{2} r_{60},$$

$$r_{60} = 1 + \frac{1}{3} r_{30} + \frac{1}{3} r_{60}.$$

Substituting the first equation into the second, we get

$$r_{60} = 1 + \frac{1}{3} (1 + \frac{1}{2} r_{60}) + \frac{1}{3} r_{60}$$

$$= \frac{4}{3} + \frac{1}{2} r_{60}.$$

Solving this gives $r_{60} = \frac{8}{3}$, hence $r_{30} = \frac{7}{3}$. Thus, the expected number of steps from $E_6$ to a recurrent state is $\frac{8}{3}$, and from $E_3$ to a recurrent state is $\frac{7}{3}$. 