Math 367 Final Exam Solutions Fall '99

1. A committee consists of 8 men and 7 women. Pick 5 at random.
   a) What is the probability you get more men than women?

   SOLUTION: We seek the probability there are at least 3 men:

$$\frac{\binom{8}{3}\binom{7}{2}+\binom{8}{4}\binom{7}{1}+\binom{8}{5}}{\binom{15}{5}}$$

   b) What is the probability you get more men than women if there are at least two women?

   SOLUTION:

$$\begin{aligned} \mathrm{P}(\geq 3 \text{ men} \,|\, \geq 2 \text{ women}) &= \frac{\mathrm{P}(\geq 3 \text{ men and } \geq 2 \text{ women})}{\mathrm{P}(\geq 2 \text{ women})} \\ &= \frac{\binom{8}{3}\binom{7}{2}/\binom{15}{5}}{[\binom{7}{5}+\binom{8}{1}\binom{7}{4}+\binom{8}{2}\binom{7}{3}+\binom{8}{3}\binom{7}{2}]/\binom{15}{5}} \\ &= \frac{\binom{8}{3}\binom{7}{2}}{\binom{7}{5}+\binom{8}{1}\binom{7}{4}+\binom{8}{2}\binom{7}{3}+\binom{8}{3}\binom{7}{2}} \end{aligned}$$

2. Find
   a) the expected value
   b) the variance

   for the number of suits in a 7-card poker hand.

   SOLUTION: Let X be the number of suits represented in the hand. Then $X = 4 - Y$, where Y is the number of suits not represented in the hand. Also, $Y = Y_1 + \cdots + Y_4$, where

$$Y_i = \begin{cases} 1 & \text{if the } i\text{-th suit is not represented in the hand} \\ 0 & \text{otherwise.} \end{cases}$$

   So

$$\mathrm{E}(Y_i) = \mathrm{P}(i\text{-th suit not in hand}) = \frac{\binom{39}{7}}{\binom{52}{7}},$$

   as there are 39 cards not in the i-th suit. Also, if $i \neq j$,

$$\begin{aligned} \mathrm{E}(Y_1 Y_j) &= \mathrm{P}(\text{neither } i\text{-th nor } j\text{-th suit in hand}) \\ &= \frac{\binom{26}{7}}{\binom{52}{7}}, \end{aligned}$$

as there are 26 cards not in either the i-th or the j-th suit.

Because $X = 4 - Y$,

$$\begin{aligned} \mathrm{E}(X) &= 4 - \mathrm{E}(Y) = 4 - [\mathrm{E}(Y_1) + \cdots + \mathrm{E}(Y_4)] \\ &= 4 - 4\frac{\binom{39}{7}}{\binom{52}{7}} = 4\left(1 - \frac{\binom{39}{7}}{\binom{52}{7}}\right). \end{aligned}$$

Also,

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{Var}(Y) = \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 \\ &= \mathrm{E}(Y_1^2) + \cdots + \mathrm{E}(Y_4^2) + \sum_{i \neq j} \mathrm{E}(Y_i Y_j) - [\mathrm{E}(Y_1) + \cdots + \mathrm{E}(Y_4)]^2 \\ &= 4\frac{\binom{39}{7}}{\binom{52}{7}} + 4 \cdot 3 \cdot \frac{\binom{26}{7}}{\binom{52}{7}} - \left[4\frac{\binom{39}{7}}{\binom{52}{7}}\right]^2. \end{aligned}$$

3. Find
   a) the expected value
   b) the variance

   for the number of aces in a 5-card poker hand.

SOLUTION: This time, $X = X_1 + \cdots + X_5$, where

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th card is an ace} \\ 0 & \text{otherwise.} \end{cases}$$

So $\mathrm{E}(X_i) = \mathrm{P}(i\text{-th card ace}) = 4/52 = 1/13$. If $i \neq j$, then

$$\begin{aligned} \mathrm{E}(X_i X_j) &= \mathrm{P}(i\text{-th and } j\text{-th are aces}) \\ &= \mathrm{P}(i\text{-th is ace} \mid j\text{-th is ace}) \cdot \mathrm{P}(j\text{-th is ace}) \\ &= \frac{3}{51} \cdot \frac{4}{52} = \frac{1}{17} \cdot \frac{1}{13}. \end{aligned}$$

Thus,

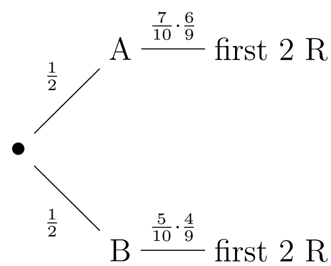
$$\mathrm{E}(X) = \mathrm{E}(X_1) + \cdots + \mathrm{E}(X_5) = 5 \cdot \frac{1}{13} = \frac{5}{13},$$

and

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \\ &= \mathrm{E}(X_1^2) + \cdots + \mathrm{E}(X_5^2) + \sum_{i \neq j} \mathrm{E}(X_i X_j) - [\mathrm{E}(X_1) + \cdots + \mathrm{E}(X_5)]^2 \\ &= 5 \cdot \frac{1}{13} + 5 \cdot 4 \cdot \frac{1}{17} \cdot \frac{1}{13} - \left[5 \cdot \frac{1}{13}\right]^2. \end{aligned}$$

4. Urn A contains 7 red balls and three green balls. Urn B contains 5 red balls and 5 green balls. Pick an urn at random and draw two balls without replacement. Suppose they both come up red.
   a) What is the probability they came from Urn A?

   Solution: Since the urn is chosen at random, the probability of choosing Urn A is $\frac{1}{2}$, and similarly for Urn B. If the balls are drawn from Urn A, the probability both are red is $\frac{7}{10} \cdot \frac{6}{9}$. If the balls are drawn from Urn B, the probability both are red is $\frac{5}{10} \cdot \frac{4}{9}$. Thus, we have the following tree diagram:

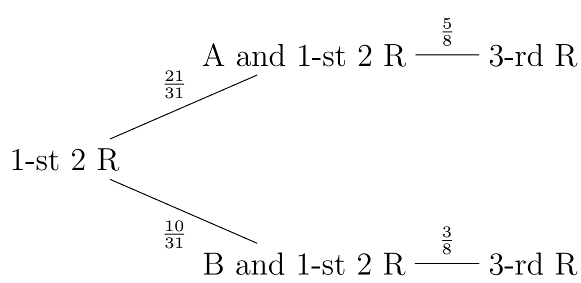


   We obtain:

$$\begin{aligned} \mathrm{P}(\text{ A} \,|\, \text{first 2 R}) &= \frac{\mathrm{P}(\text{ A and first 2 R})}{\mathrm{P}(\text{first 2 R})} \\ &= \frac{\frac{1}{2} \cdot \frac{7}{10} \cdot \frac{6}{9}}{\frac{1}{2} \cdot \frac{7}{10} \cdot \frac{6}{9} + \frac{1}{2} \cdot \frac{5}{10} \cdot \frac{4}{9}} \\ &= \frac{42}{62} = \frac{21}{31}. \end{aligned}$$

   b) If a third ball is drawn from the same urn (without replacement), what is the probability it is red?

   Solution: We now make a conditional tree diagram, starting with the condition that the first two balls are red:



   Thus,

$$\mathrm{P}(\text{3-rd R} \,|\, \text{1-st 2 R}) = \frac{21}{31} \cdot \frac{5}{8} + \frac{10}{31} \cdot \frac{3}{8}.$$

5. Flip three coins. Then draw letters with replacement from a word, to be detemined below, until you get a vowel.

   If you get no heads, draw from APPLE.
   If you get one head, draw from PEAR.
   If you get two heads, draw from PLUM.
   If you get three heads, draw from KUMQUAT.

   How many letters do you expect to draw?

   SOLUTION: This is a conditional expected value problem. For whichever word is used, the conditional expected value for the number of draws from that word is $\frac{1}{p}$, where p is the probability of drawing a vowel in a single draw from that word. We obtain

$$\begin{aligned} \mathrm{E}(X) &= \mathrm{E}(X \mid 0\mathrm{H}) \cdot \mathrm{P}(0\mathrm{H}) + \mathrm{E}(X \mid 1\mathrm{H}) \cdot \mathrm{P}(1\mathrm{H}) + \mathrm{E}(X \mid 2\mathrm{H}) \cdot \mathrm{P}(2\mathrm{H}) + \mathrm{E}(X \mid 3\mathrm{H}) \cdot \mathrm{P}(3\mathrm{H}) \\ &= \frac{1}{(2/5)} \cdot \frac{1}{8} + \frac{1}{(1/2)} \cdot \frac{3}{8} + \frac{1}{(1/4)} \cdot \frac{3}{8} + \frac{1}{(3/7)} \cdot \frac{1}{8}. \end{aligned}$$

6. Suppose that

$$f(z) = \frac{4}{(3-z)^2}$$

   is the generating function for a random variable X. Find

   a) $\mathrm{P}(X > 0)$ b) $\mathrm{P}(X > 1)$ c) $\mathrm{E}(X)$ d) $\mathrm{Var}(X)$

   SOLUTION: We will need $f'(z)$ and $f''(z)$, and we may as well calculate them now. We have $f(z) = 4(3-z)^{-2}$.

$$\begin{aligned} f'(z) &= 4(-2)(3-z)^{-3}(-1) = 8(3-z)^{-3} = \frac{8}{(3-z)^3} \\ f''(z) &= 8(-3)(3-z)^{-4}(-1) = 24(3-z)^{-4} = \frac{24}{(3-z)^4}. \end{aligned}$$

   For a), we have

$$\mathrm{P}(X > 0) = 1 - \mathrm{P}(X = 0) = 1 - f(0) = 1 - \frac{4}{9} = \frac{5}{9}.$$

   For b), we have

$$\begin{aligned} \mathrm{P}(X > 1) &= 1 - [\mathrm{P}(X = 0) + \mathrm{P}(X = 1)] = 1 - [f(0) + f'(0)] \\ &= 1 - \left[\frac{4}{9} + \frac{8}{27}\right] = 1 - \frac{20}{27} = \frac{7}{27}. \end{aligned}$$

For c), we have

$$\mathrm{E}(X) = f'(1) = \frac{8}{8} = 1.$$

For d), we have

$$\mathrm{Var}(X) = f''(1) + f'(1) - (f'(1))^2 = \frac{24}{16} + 1 - 1^2 = \frac{3}{2}.$$

7. Consider the transition matrix

$$\begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

a) Find R_i for each state $i = 1, \ldots, 8$.

SOLUTION:

$$\begin{aligned} R_1 &= \{E_1, E_4, E_6\} \\ R_2 &= \{E_2, E_3, E_4, E_5, E_7, E_1, E_6, E_8\} \\ R_3 &= \{E_2, E_3, E_4, E_5, E_7, E_1, E_6, E_8\} \\ R_4 &= \{E_4, E_6, E_1\} \\ R_5 &= \{E_5, E_8\} \\ R_6 &= \{E_1, E_4, E_6\} \\ R_7 &= \{E_1, E_5, E_7, E_4, E_6, E_8\} \\ R_8 &= \{E_5, E_8\}. \end{aligned}$$

b) Classify the recurrent and the transient states.

SOLUTION: E_2, E_3, and E_7 are transient, as E_1 is in R_2, R_3, and R_7, but none of E_2, E_3, and E_7 is in R_1. The remaining states (E_1, E_4, E_5, E_6, and E_8) are recurrent.

c) If you start in E_3, what is the probability you will return there?

SOLUTION: We wish to find h_{33}. The general formula is

$$h_{ij} = p_{ij} + \sum_{k \neq j} p_{ik} h_{kj}.$$

Thus,

$$\begin{aligned} h_{33} &= 0 + \frac{1}{4} h_{23} + \frac{1}{4} h_{43} + \frac{1}{4} h_{53} + \frac{1}{4} h_{73} \\ &= \frac{1}{4} h_{23}. \end{aligned}$$

Here, we've used that E_3 is not in R_4, R_5, or R_7, so $h_{43} = h_{53} = h_{73} = 0$. Thus, we must calculate h_{23}.

$$\begin{aligned} h_{23} &= \frac{1}{4} + \frac{1}{4} h_{23} + \frac{1}{2} h_{43} \\ &= \frac{1}{4} + \frac{1}{4} h_{23}, \end{aligned}$$

as $h_{43} = 0$. Collecting terms, we see $\frac{3}{4} h_{23} = \frac{1}{4}$, so $h_{23} = \frac{1}{3}$. Thus, $h_{33} = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{12}$.

d) If you start in E_2, how many times do you expect to visit E_3?

SOLUTION: This time, we want v_{23}. We have

$$v_{23} = \frac{h_{23}}{1 - h_{33}} = \frac{(1/3)}{1 - (1/12)} = \frac{4}{11}.$$

e) For each transient state, compute the expected number of steps needed to reach a recurrent state.

SOLUTION: We combine the recurrent states into a single absorbing state, E_0. The new transition matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

We wish to find r_{20}, r_{30} and r_{70} for this new matrix. The general formula is $r_{ij} = 1 + \sum_{k \neq j} p_{ik} r_{kj}$. Thus,

$$r_{20} = 1 + \frac{1}{4} r_{20} + \frac{1}{4} r_{30}$$
$$r_{30} = 1 + \frac{1}{4} r_{20} + \frac{1}{4} r_{70}$$
$$r_{70} = 1 + \frac{1}{2} r_{70}.$$

The last equation may be immediately solved: $r_{70} = 2$. Substituting this into the second equation, we get $r_{30} = \frac{3}{2} + \frac{1}{4} r_{20}$. Collecting terms in the first two equations, we get

$$\frac{3}{4} r_{20} - \frac{1}{4} r_{30} = 1$$
$$-\frac{1}{4} r_{20} + r_{30} = \frac{3}{2}.$$

Multiplying the first equation by 4 and adding, we get

$$\frac{11}{4} r_{20} = \frac{11}{2},$$

so $r_{20} = 2$. Thus, $r_{30} = \frac{3}{2} + \frac{1}{4} \cdot 2 = 2$. So for any one of the transient states, it takes about two steps to reach a recurrent state.