1. Fred goes to the casino with $100. He decides he’s going to keep betting, $1 at a time for even money, on a game where he has a probability of .47 of winning, until his money runs out. How long can he expect to play?

**Solution:** The casino has so much money that we may as well assume it (not Fred) has unlimited credit. So we cast the casino as Peter and Fred as Paul. Thus, \( p = .53 \) and \( q = .47 \). Because \( p > q \), the probability “Peter” gets ahead by $100 is 1. The expected duration of play is

\[
\frac{100}{p-q} = \frac{100}{.06}.
\]

2. Peter bets $1 at a time for even money on a game he has a probability of .41 of winning. If you give him unlimited credit, what is the probability he’ll ever get $10 ahead?

**Solution:** Here, \( p < q \), so the probability of getting ahead by one stake is \( \frac{p}{q} = \frac{41}{59} \). Thus, the probability of getting ahead by 10 stakes is \( \left( \frac{41}{59} \right)^{10} \).

3. Consider the transition matrix

\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Sample Questions

a) Find $R_i$ for each state $i = 1, \ldots, 8$.

Solution:

\[ R_1 = \{ E_1, E_2, E_3, E_6, E_7 \} \]
\[ R_2 = \{ E_1, E_2, E_3, E_6, E_7 \} \]
\[ R_3 = \{ E_3, E_6, E_7 \} \]
\[ R_4 = \{ E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8 \} \]
\[ R_5 = \{ E_5, E_8 \} \]
\[ R_6 = \{ E_3, E_6, E_7 \} \]
\[ R_7 = \{ E_3, E_6, E_7 \} \]
\[ R_8 = \{ E_5, E_8 \} \]

b) Classify the recurrent and the transient states.

Solution: $E_1$, $E_2$, and $E_4$ are transient, because $E_3$ is in each of $R_1$, $R_2$, and $R_4$, but $E_1$, $E_2$, and $E_4$ are not in $R_3$. The rest of the states ($E_3$, $E_5$, $E_6$, $E_7$, and $E_8$) are recurrent.

c) Compute $h_{43}$.

Solution: We use the formula $h_{ij} = p_{ij} + \sum_{k \neq j} p_{ik} h_{kj}$. This gives $h_{43} = 0 + \frac{1}{4} h_{23} + \frac{1}{4} h_{43} + \frac{1}{2} h_{53}$. Since $E_3 \notin R_5$, $h_{53} = 0$, and we get

\[ \frac{3}{4} h_{43} = \frac{1}{4} h_{23}, \]

hence $h_{43} = \frac{1}{3} h_{23}$. Thus, we must compute $h_{23}$. We can see from principles that $h_{23} = 1$, as $\{ E_3, E_6, E_7 \}$ is the only recurrent cycle you can get to from $E_2$. We also compute it directly: To directly compute $h_{23}$, we’ll also need $h_{13}$. The general formula gives

\[ h_{23} = \frac{1}{3} + \frac{1}{3} h_{13} + \frac{1}{3} h_{23} \]
\[ h_{13} = 0 + \frac{1}{2} h_{13} + \frac{1}{2} h_{23}, \]
Sample Questions

giving
\[ \frac{1}{3} = -\frac{1}{3}h_{13} + \frac{2}{3}h_{23} \]
\[ 0 = \frac{1}{2}h_{13} - \frac{1}{2}h_{23}. \]

Multiplying the first equation by 3 and the second by 2 and adding gives \( h_{23} = 1 \). (The second equation gives \( h_{13} = h_{23} \), if we care.) Thus,

\[ h_{43} = \frac{1}{3}h_{23} = \frac{1}{3}. \]

\[ d) \text{ Compute } h_{45}. \]

\textbf{Solution:} Again, we can argue from principles. If you start in \( E_4 \), you must wind up in either \( \{E_3, E_6, E_7\} \) or in \( \{E_5, E_8\} \), and you cannot visit both. Thus, \( h_{43} + h_{45} = 1 \), so \( h_{45} = \frac{2}{3} \).

We can also compute it directly:

\[ h_{45} = \frac{1}{2} + \frac{1}{4}h_{25} + \frac{1}{4}h_{45}. \]

Since \( E_5 \notin R_2, h_{25} = 0 \), so \( \frac{3}{4}h_{45} = \frac{1}{2} \). So \( h_{45} = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3} \).

\[ e) \text{ Compute } v_{12}. \]

\textbf{Solution:} The basic formula is

\[ v_{12} = \frac{h_{12}}{1 - h_{22}}. \]

To calculate \( h_{12} \) and \( h_{22} \), we have

\[ h_{12} = \frac{1}{2} + \frac{1}{2}h_{12} \]
\[ h_{22} = \frac{1}{3} + \frac{1}{3}h_{12} + \frac{1}{3}h_{32}. \]

The first equation gives \( \frac{1}{2}h_{12} = \frac{1}{2} \), so \( h_{12} = 1 \). Since \( E_2 \notin R_3, h_{32} = 0 \). Thus, the second equation gives \( h_{22} = \frac{1}{3} + \frac{1}{3}h_{12} = \frac{2}{3} \).

Thus,

\[ v_{12} = \frac{h_{12}}{1 - h_{22}} = \frac{1}{\left(\frac{1}{3}\right)} = 3. \]
Sample Questions

f) Compute $r_{13}$ and $r_{23}$.

Solution: The formula here is $r_{ij} = 1 + \sum_{k \neq j} p_{ik} r_{kj}$. Thus,

\[
\begin{align*}
    r_{13} &= 1 + \frac{1}{2} r_{13} + \frac{1}{2} r_{23} \\
    r_{23} &= 1 + \frac{1}{3} r_{13} + \frac{1}{3} r_{23}.
\end{align*}
\]

This gives

\[
\begin{align*}
    \frac{1}{2} r_{13} - \frac{1}{2} r_{23} &= 1 \\
    -\frac{1}{3} r_{13} + \frac{2}{3} r_{23} &= 1.
\end{align*}
\]

Multiplying the first equation by 2 and the second by 3 and adding, we get $r_{23} = 5$. The first equation now gives $r_{13} = 7$.

g) For each transient state, compute the expected number of steps needed to reach a recurrent state.

Solution: We combine all the recurrent states into one single absorbing state, $E_0$. If we order the states as $E_0, E_1, E_2, E_4$, the resulting transition matrix is

\[
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & \frac{1}{2} & \frac{1}{2} & 0 \\
    \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
    \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4}
\end{bmatrix}
\]

If $i = 1, 2, 4$, the expected number of steps to reach a recurrent state from $E_i$ in the original problem is equal to $r_{i0}$ for this new matrix. We have

\[
\begin{align*}
    r_{10} &= 1 + \frac{1}{2} r_{10} + \frac{1}{2} r_{20} \\
    r_{20} &= 1 + \frac{1}{3} r_{10} + \frac{1}{3} r_{20} \\
    r_{40} &= 1 + \frac{1}{4} r_{20} + \frac{1}{4} r_{40}.
\end{align*}
\]
The first two rows give
\[ \frac{1}{2} r_{10} - \frac{1}{2} r_{20} = 1 \]
\[ -\frac{1}{3} r_{10} + \frac{2}{3} r_{20} = 1. \]

These are a relabeled version of the equations we had in part f) (for a good reason which might not repeat itself if the problem were altered), so the same solution holds: \( r_{20} = 5 \) and \( r_{10} = 7 \).

The third equation now gives
\[ \frac{3}{4} r_{40} = 1 + \frac{1}{4} r_{20} = 1 + \frac{5}{4}, \]
so \( r_{40} = \frac{4}{3} + \frac{5}{3} = 3. \)