

1. Fred goes to the casino with \$100. He decides he's going to keep betting, \$1 at a time for even money, on a game where he has a probability of .47 of winning, until his money runs out. How long can he expect to play?

SOLUTION: The casino has so much money that we may as well assume it (not Fred) has unlimited credit. So we cast the casino as Peter and Fred as Paul. Thus,  $p = .53$  and  $q = .47$ . Because  $p > q$ , the probability "Peter" gets ahead by \$100 is 1. The expected duration of play is

$$\frac{100}{p - q} = \frac{100}{.06}.$$

2. Peter bets \$1 at a time for even money on a game he has a probability of .41 of winning. If you give him unlimited credit, what is the probability he'll ever get \$10 ahead?

SOLUTION: Here,  $p < q$ , so the probability of getting ahead by one stake is  $\frac{p}{q} = \frac{41}{59}$ . Thus, the probability of getting ahead by 10 stakes is  $(\frac{41}{59})^{10}$ .

3. Consider the transition matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

**Sample Questions**

a) Find  $R_i$  for each state  $i = 1, \dots, 8$ .

SOLUTION:

$$R_1 = \{E_1, E_2, E_3, E_6, E_7\}$$

$$R_2 = \{E_1, E_2, E_3, E_6, E_7\}$$

$$R_3 = \{E_3, E_6, E_7\}$$

$$R_4 = \{E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8\}$$

$$R_5 = \{E_5, E_8\}$$

$$R_6 = \{E_3, E_6, E_7\}$$

$$R_7 = \{E_3, E_6, E_7\}$$

$$R_8 = \{E_5, E_8\}$$

b) Classify the recurrent and the transient states.

SOLUTION:  $E_1, E_2$ , and  $E_4$  are transient, because  $E_3$  is in each of  $R_1, R_2$ , and  $R_4$ , but  $E_1, E_2$ , and  $E_4$  are not in  $R_3$ . The rest of the states ( $E_3, E_5, E_6, E_7$ , and  $E_8$ ) are recurrent.

c) Compute  $h_{43}$ .

SOLUTION: We use the formula  $h_{ij} = p_{ij} + \sum_{k \neq j} p_{ik} h_{kj}$ . This gives  $h_{43} = 0 + \frac{1}{4}h_{23} + \frac{1}{4}h_{43} + \frac{1}{2}h_{53}$ . Since  $E_3 \notin R_5$ ,  $h_{53} = 0$ , and we get

$$\frac{3}{4}h_{43} = \frac{1}{4}h_{23},$$

hence  $h_{43} = \frac{1}{3}h_{23}$ . Thus, we must compute  $h_{23}$ . We can see from principles that  $h_{23} = 1$ , as  $\{E_3, E_6, E_7\}$  is the only recurrent cycle you can get to from  $E_2$ . We also compute it directly: To directly compute  $h_{23}$ , we'll also need  $h_{13}$ . The general formula gives

$$h_{23} = \frac{1}{3} + \frac{1}{3}h_{13} + \frac{1}{3}h_{23}$$

$$h_{13} = 0 + \frac{1}{2}h_{13} + \frac{1}{2}h_{23},$$

giving

$$\begin{aligned}\frac{1}{3} &= -\frac{1}{3}h_{13} + \frac{2}{3}h_{23} \\ 0 &= \frac{1}{2}h_{13} - \frac{1}{2}h_{23}.\end{aligned}$$

Multiplying the first equation by 3 and the second by 2 and adding gives  $h_{23} = 1$ . (The second equation gives  $h_{13} = h_{23}$ , if we care.) Thus,

$$h_{43} = \frac{1}{3}h_{23} = \frac{1}{3}.$$

d) Compute  $h_{45}$ .

SOLUTION: Again, we can argue from principles. If you start in  $E_4$ , you must wind up in either  $\{E_3, E_6, E_7\}$  or in  $\{E_5, E_8\}$ , and you cannot visit both. Thus,  $h_{43} + h_{45} = 1$ , so  $h_{45} = \frac{2}{3}$ . We can also compute it directly:

$$h_{45} = \frac{1}{2} + \frac{1}{4}h_{25} + \frac{1}{4}h_{45}.$$

Since  $E_5 \notin R_2$ ,  $h_{25} = 0$ , so  $\frac{3}{4}h_{45} = \frac{1}{2}$ . So  $h_{45} = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3}$ .

e) Compute  $v_{12}$ .

SOLUTION: The basic formula is

$$v_{12} = \frac{h_{12}}{1 - h_{22}}.$$

To calculate  $h_{12}$  and  $h_{22}$ , we have

$$\begin{aligned}h_{12} &= \frac{1}{2} + \frac{1}{2}h_{12} \\ h_{22} &= \frac{1}{3} + \frac{1}{3}h_{12} + \frac{1}{3}h_{32}.\end{aligned}$$

The first equation gives  $\frac{1}{2}h_{12} = \frac{1}{2}$ , so  $h_{12} = 1$ . Since  $E_2 \notin R_3$ ,  $h_{32} = 0$ . Thus, the second equation gives  $h_{22} = \frac{1}{3} + \frac{1}{3}h_{12} = \frac{2}{3}$ . Thus,

$$v_{12} = \frac{h_{12}}{1 - h_{22}} = \frac{1}{\left(\frac{1}{3}\right)} = 3.$$

*Sample Questions*

f) Compute  $r_{13}$  and  $r_{23}$ .

SOLUTION: The formula here is  $r_{ij} = 1 + \sum_{k \neq j} p_{ik} r_{kj}$ . Thus,

$$\begin{aligned} r_{13} &= 1 + \frac{1}{2}r_{13} + \frac{1}{2}r_{23} \\ r_{23} &= 1 + \frac{1}{3}r_{13} + \frac{1}{3}r_{23}. \end{aligned}$$

This gives

$$\begin{aligned} \frac{1}{2}r_{13} - \frac{1}{2}r_{23} &= 1 \\ -\frac{1}{3}r_{13} + \frac{2}{3}r_{23} &= 1. \end{aligned}$$

Multiplying the first equation by 2 and the second by 3 and adding, we get  $r_{23} = 5$ . The first equation now gives  $r_{13} = 7$ .

g) For each transient state, compute the expected number of steps needed to reach a recurrent state.

SOLUTION: We combine all the recurrent states into one single absorbing state,  $E_0$ . If we order the states as  $E_0, E_1, E_2, E_4$ , the resulting transition matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

If  $i = 1, 2, 4$ , the expected number of steps to reach a recurrent state from  $E_i$  in the original problem is equal to  $r_{i0}$  for this new matrix. We have

$$\begin{aligned} r_{10} &= 1 + \frac{1}{2}r_{10} + \frac{1}{2}r_{20} \\ r_{20} &= 1 + \frac{1}{3}r_{10} + \frac{1}{3}r_{20} \\ r_{40} &= 1 + \frac{1}{4}r_{20} + \frac{1}{4}r_{40}. \end{aligned}$$

The first two rows give

$$\begin{aligned}\frac{1}{2}r_{10} - \frac{1}{2}r_{20} &= 1 \\ -\frac{1}{3}r_{10} + \frac{2}{3}r_{20} &= 1.\end{aligned}$$

These are a relabeled version of the equations we had in part f) (for a good reason which might not repeat itself if the problem were altered), so the same solution holds:  $r_{20} = 5$  and  $r_{10} = 7$ .

The third equation now gives

$$\frac{3}{4}r_{40} = 1 + \frac{1}{4}r_{20} = 1 + \frac{5}{4},$$

$$\text{so } r_{40} = \frac{4}{3} + \frac{5}{3} = 3.$$