**Math 367 Exam 3 Solutions Fall '99**

1. Suppose that

$$f(z) = \frac{1}{(2-z)^2}$$

is the generating function for a random variable $X$. Find

a) $\mathrm{P}(X > 0)$ b) $\mathrm{P}(X > 1)$ c) $\mathrm{E}(X)$ d) $\mathrm{Var}(X)$

SOLUTION: We have $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_i z^i + \cdots$, where $a_i = \mathrm{P}(X = i)$. Thus,

$$\mathrm{P}(X = 0) = a_0 = f(0) = \frac{1}{2^2} = \frac{1}{4},$$

so

$$\mathrm{P}(X > 0) = 1 - \mathrm{P}(X = 0) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Also, the calculus of power series tells us $a_1 = f'(0)$. Differentiating the formula for $f$ gives

$$f'(z) = -2(2-z)^{-3}(-1) = \frac{2}{(2-z)^3},$$

so $\mathrm{P}(X = 1) = a_1 = f'(0) = \frac{2}{2^3} = \frac{1}{4}$. We see

$$\mathrm{P}(X > 1) = 1 - [\mathrm{P}(X = 0) + \mathrm{P}(X = 1)] = 1 - \left[\frac{1}{4} + \frac{1}{4}\right] = \frac{1}{2}.$$

The expected value is given by $f'(1)$:

$$\mathrm{E}(X) = f'(1) = \frac{2}{(2-1)^3} = 2.$$

We may compute the variance from this and the second derivative of $f$. The latter is given by

$$f''(z) = 2(-3)(2-z)^{-4}(-1) = \frac{6}{(2-z)^4}.$$

We obtain

$$\begin{aligned}\mathrm{Var}(X) &= f''(1) + f'(1) - (f'(1))^2 \\ &= \frac{6}{1^4} + \frac{2}{1^3} - \left(\frac{2}{1^3}\right)^2 = 4.\end{aligned}$$

2. George goes to the casino with \$200. He decides to keep betting, \$1 at a time for even money, on a game where he has a probability of .45 of winning, until his money runs out. How long can he expect to play?

   SOLUTION: Here, "Peter" is the casino, and has unlimited credit. $p = .55$, $q = .45$, and the expected number of plays for Peter to be ahead by 200 is

$$\frac{200}{p-q} = \frac{200}{.1} = 2000.$$

3. Bob goes to the casino with \$300. He will bet \$1 at a time, with a probability of .47 of winning. He will leave if he ever gets \$50 ahead, or, alternatively, if he goes broke.
   a) What is the probability Bob gets the \$50?
   b) How many bets can he expect to make?

   SOLUTION: We'll take Bob to be Peter and the casino to be Paul. So $p = .47$ and $q = .53$. The most the casino will stake (net) is \$50, and the most Bob will stake is \$300, so this is Gambler's Ruin, with $t = 350$ and $s = 300$. The probability Bob wins is

$$p^* = \frac{1-r^{300}}{1-r^{350}} = \frac{1-(\frac{53}{47})^{300}}{1-(\frac{53}{47})^{350}}.$$

   The expected number of plays is

$$\frac{p^* \cdot 350 - 300}{.47 - .53}$$

   where $p^*$ is the probability Bob wins.

4. Albert bets \$1 at a time for even money on a game he has a probability of .43 of winning. If you give him unlimited credit, what is the probability he'll ever get \$10 ahead?

   SOLUTION: Here, $p = .43$ and $q = .57$. The probability Albert ever gets \$10 ahead is

$$\left(\frac{p}{q}\right)^{10} = \left(\frac{43}{57}\right)^{10}.$$

5. Consider the transition matrix

$$\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

a) Find $R_i$ for each state $i = 1, \ldots, 8$.

SOLUTION: Here,

$$\begin{aligned} R_1 &= \{E_1, E_2, E_3, E_8, E_6\} \\ R_2 &= \{E_1, E_2, E_3, E_8, E_6\} \\ R_3 &= \{E_3, E_8, E_6\} \\ R_4 &= \{E_4, E_7\} \\ R_5 &= \{E_3, E_4, E_5, E_8, E_6, E_7\} \\ R_6 &= \{E_6, E_8, E_3\} \\ R_7 &= \{E_4, E_7\} \\ R_8 &= \{E_3, E_8, E_6\} \end{aligned}$$

b) Classify the recurrent and the transient states.

SOLUTION: $E_1$, $E_2$ and $E_5$ are transitive, as $E_3$ lies in all of $R_1$, $R_2$ and $R_5$, but none of $E_1$, $E_2$, and $E_5$ lies in $R_3$.

The remaining states, $E_3$, $E_4$, $E_6$, $E_7$, and $E_8$, are recurrent.

c) If you start in $E_2$, what is the probability you will return there?

SOLUTION: Here, we seek

$$\begin{aligned} h_{22} &= p_{22} + \sum_{k \neq 2} p_{2k} h_{k2} \\ &= \frac{1}{4} + \frac{1}{4} h_{12} + \frac{1}{4} h_{32} + \frac{1}{4} h_{82}. \end{aligned}$$

Since $E_2$ lies in neither $R_3$ nor $R_8$, $h_{32} = h_{82} = 0$, so

$$h_{22} = \frac{1}{4} + \frac{1}{4} h_{12}.$$

Thus, we need to find

$$\begin{aligned} h_{12} &= p_{12} + \sum_{k \neq 2} p_{1k} h_{k2} \\ &= \frac{1}{4} + \frac{3}{4} h_{12}. \end{aligned}$$

We can solve this immediately, getting

$$\frac{1}{4} h_{12} = \frac{1}{4},$$

so $h_{12} = 1$. Substituting this into the equation for $h_{22}$, we get

$$h_{22} = \frac{1}{4} + \frac{1}{4} h_{12} = \frac{1}{2}.$$

d) If you start in $E_1$, how many times do you expect to visit $E_2$?

SOLUTION: Here, we seek

$$v_{12} = \frac{h_{12}}{1 - h_{22}} = \frac{1}{1 - \frac{1}{2}} = 2$$

by the calculations in part c).

e) If you start in $E_1$, how long do you expect it to take you to reach $E_2$?

SOLUTION: We wish to find

$$\begin{aligned} r_{12} &= 1 + \sum_{k \neq 2} p_{1k} r_{k2} \\ &= 1 + \frac{3}{4} r_{12}. \end{aligned}$$

So $\frac{1}{4} r_{12} = 1$, hence $r_{12} = 4$.

f) For each transient state, compute the expected number of steps needed to reach a recurrent state.

Solution: We combine all five recurrent states into one absorbing state, $r_0$. The new transition matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

We wish to solve the following equations:

$$\begin{aligned} r_{10} &= 1 + \frac{3}{4}r_{10} + \frac{1}{4}r_{20} \\ r_{20} &= 1 + \frac{1}{4}r_{10} + \frac{1}{4}r_{20} \\ r_{50} &= 1 + \frac{1}{4}r_{50}. \end{aligned}$$

The last equation can be solved alone: $\frac{3}{4}r_{50} = 1$, so $r_{50} = \frac{4}{3}$. Thus, it takes about $\frac{4}{3}$ steps to reach a recurrent state from $E_5$.

The first two equations give

$$\begin{aligned} \frac{1}{4}r_{10} - \frac{1}{4}r_{20} &= 1 \\ -\frac{1}{4}r_{10} + \frac{3}{4}r_{20} &= 1 \end{aligned}$$

Adding these up, we get $\frac{1}{2}r_{20} = 2$, so $r_{20} = 4$. The first equation now gives $\frac{1}{4}r_{10} = 1 + \frac{1}{4}r_{20} = 1 + 1 = 2$, so $r_{10} = 8$. Thus, it takes about 4 steps to get from $E_2$ to a recurrent state, and about 8 steps to get from $E_1$ to a recurrent state.