Math 367 **Exam 3 Answers** **Spring '04**

1. Suppose that

$$f(z) = \frac{6}{(7-z)}$$

is the generating function for a random variable X. Find

a) $\mathrm{P}(X > 2)$ b) $\mathrm{E}(X)$ c) $\mathrm{Var}(X)$

SOLUTION: $f'(z) = 6(7-x)^{-2}$, $f''(z) = 12(7-x)^{-3}$. $\mathrm{E}(X) = \frac{1}{6}$. $\mathrm{Var}(X) = \frac{1}{18} + \frac{1}{6} - \frac{1}{36}$. $\mathrm{P}(X > 2) = 1 - [\frac{6}{7} + \frac{6}{49} + \frac{6}{7^3}]$.

2. Consider the transition matrix

$$\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Find R_i for each state $i = 1, \ldots, 8$. SOLUTION: $R_1 = R_5 = \{E_1, E_5\}$. $R_2 = R_3 = \{E_1, E_2, E_3, E_5\}$. $R_4 = R_8 = \{E_4, E_8\}$. $R_6 = \{\text{all states}\}$. $R_7 = \{E_7\}$.

b) Classify the recurrent and the transient states. SOLUTION: E_2, E_3, E_6 transitive.

c) If you start in E_3, what is the probability you will return there? SOLUTION:

$$h_{33} = \frac{1}{2}h_{23} + \frac{1}{2}h_{53} = \frac{1}{2}h_{23}$$
$$h_{23} = \frac{1}{2} + \frac{1}{4}h_{13} + \frac{1}{4}h_{53} = \frac{1}{2}, \text{ so } h_{33} = \frac{1}{4}.$$

d) If you start in E_6, how many times do you expect to visit E_3? SOLUTION:

$$h_{63} = \frac{1}{4}h_{23} + \frac{1}{4}h_{63} + \frac{1}{4}h_{73} + \frac{1}{4}h_{83} = \frac{1}{8} + \frac{1}{4}h_{63} = \frac{1}{6}.$$

$$v_{63} = \frac{h_{63}}{1 - h_{33}} = \frac{\frac{1}{6}}{\frac{3}{4}} = \frac{2}{9}.$$

e) If you start in E_3, how long do you expect it to take you to reach E_1? SOLUTION:

$$r_{31} = 1 + \frac{1}{2}r_{21} + \frac{1}{2}r_{51} = 2 + \frac{1}{2}\left(\frac{3}{2} + \frac{1}{2}r_{31}\right) = \frac{11}{4} + \frac{1}{4}r_{31} = \frac{11}{3}$$

$$r_{21} = 1 + \frac{1}{2}r_{31} + \frac{1}{4}r_{51} = \frac{3}{2} + \frac{1}{2}r_{31}$$

$$r_{51} = 1 + \frac{1}{2}r_{51} = 2.$$

f) From each transient state, how long do you expect it take you to reach a recurrent state? SOLUTION: We combine the recurrent states into a single state E_*, listed fourth in the following transition matrix:

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{2*} = 1 + \frac{1}{2}r_{3*} = 1 + \frac{1}{2}\left(1 + \frac{1}{2}r_{2*}\right) = \frac{3}{2} + \frac{1}{4}r_{2*} = 2$$

$$r_{3*} = 1 + \frac{1}{2}r_{2*} = 2$$

$$r_{6*} = 1 + \frac{1}{4}r_{2*} + \frac{1}{4}r_{6*} = \frac{3}{2} + \frac{1}{4}r_{6*} = 2.$$

3. You roll a die until you get the sequence 1,1,2,2. What is the transition matrix for the Markov system describing this game?
SOLUTION:

$$\begin{bmatrix} \frac{5}{6} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{5}{6} & 0 & \frac{1}{6} & 0 & 0 \\ \frac{4}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{4}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Find a fixed probability vector for the transition matrix

$$\begin{bmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

SOLUTION: $v = \begin{bmatrix} \frac{2}{8} & \frac{3}{8} & \frac{3}{8} \end{bmatrix}$.