Math 367 Exam 3 Solutions Fall '02

1. Suppose that

$$f(z) = \frac{4}{(5-z)}$$

is the generating function for a random variable X. Find
a) $\mathrm{P}(X > 1)$ b) $\mathrm{E}(X)$ c) $\mathrm{Var}(X)$

Solution: We have $f(z) = 4(5-z)^{-1}$, so $f'(z) = 4(5-z)^{-2}$ and $f''(z) = 8(5-z)^{-3}$. Thus,

$$\begin{aligned}\mathrm{P}(X > 1) &= 1 - [P(X=0) + P(X=1)] \\ &= 1 - [f(0) + f'(0)] = 1 - \left[\frac{4}{5} + \frac{4}{25}\right] = \frac{1}{25},\end{aligned}$$

and $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ are calculated as follows:

$$\mathrm{E}(X) = f'(1) = \frac{4}{16} = \frac{1}{4}.$$

$$\mathrm{Var}(X) = f''(1) + f'(1) - (f'(1))^2 = \frac{8}{64} + \frac{1}{4} - \left(\frac{1}{4}\right)^2 = \frac{5}{16}.$$

2. Bob goes to the casino with \$100. He will bet \$1 at a time, with a probability of .43 of winning. He will leave if he ever gets \$25 ahead, or, alternatively, if he goes broke.
a) What is the probability Bob gets the \$25?
b) How many bets can he expect to make?

Solution: This is Gambler's Ruin with $t = 125$ and $s = 100$. Here $p = .43$, $q = .57$, and $r = \frac{57}{43}$. So the probability Bob gets the \$25 is

$$p^* = \frac{1 - \left(\frac{57}{43}\right)^{100}}{1 - \left(\frac{57}{43}\right)^{125}},$$

and the expected duration of play is

$$\frac{p^* t - s}{p - q} = \frac{125p^* - 100}{-.14},$$

with p^* as above.

3. Albert bets \$1 at a time for even money on a game he has a probability of .48 of winning. If you give him unlimited credit, what is the probability he'll ever get \$12 ahead?

   SOLUTION: Since $p < q$, the probability is $(\frac{p}{q})^{12} = (\frac{48}{52})^{12}$.

4. Consider the transition matrix

$$\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

   a) Find R_i for each state $i = 1, \ldots, 8$.

      SOLUTION: $R_1 = \{E_1, E_5\}$.
      $R_2 = \{E_1, E_3, E_6, E_5, E_2, E_4, E_8\}$.
      $R_3 = R_2$.
      $R_4 = \{E_4, E_6, E_8\}$.
      $R_5 = \{E_1, E_5\}$.
      $R_6 = \{E_4, E_6, E_8\}$.
      $R_7 = \{E_7, E_8, E_6, E_4\}$.
      $R_8 = \{E_6, E_4, E_8\}$.

   b) Classify the recurrent and the transient states.

      SOLUTION: $\{E_1, E_5\}$ and $\{E_4, E_6, E_8\}$ are recurrent clusters. E_2, E_3 and E_7 are transient, because E_8 is in all of R_2, R_3 and R_7 but none of E_2, E_3 and E_7 are in R_8.

   c) If you start in E_2, what is the probability you will return there?

      SOLUTION:

$$h_{22} = \frac{1}{4}h_{12} + \frac{1}{2}h_{32} + \frac{1}{4}h_{62} = \frac{1}{2}h_{32},$$

      as E_2 cannot be reached from E_1 or E_6.

$$h_{32} = \frac{1}{2} + \frac{1}{2}h_{52} = \frac{1}{2},$$

      as E_2 cannot be reached from E_5. So $h_{22} = \frac{1}{4}$.

d) If you start in E_3, how many times do you expect to visit E_2?

SOLUTION: $v_{32} = \frac{h_{32}}{1-h_{22}} = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$.

e) If you start in E_7, how long do you expect it to take you to reach E_4?

SOLUTION:

$$r_{74} = 1 + \frac{3}{4}r_{74} + \frac{1}{4}r_{84}$$
$$r_{84} = 1 + 1 \cdot r_{64}$$
$$r_{64} = 1 + \frac{1}{2}r_{84}.$$

Substituting this last into the preceding line, we get

$$r_{84} = 1 + 1 + \frac{1}{2}r_{84}, \quad \text{hence}$$
$$\frac{1}{2}r_{84} = 2.$$

Substituting $r_{84} = 4$ in the very first line, we get

$$\frac{1}{4}r_{74} = 2,$$

so $r_{74} = 8$.

f) If you start in E_2, how long do you expect it take you to reach a recurrent state?

SOLUTION: We combine the recurrent states into one absorbing state E_*, obtaining a new transition matrix, whose rows and columns correspond to E_2, E_3, E_7, E_*:

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{2*} = 1 + \frac{1}{2}r_{3*}$$
$$r_{3*} = 1 + \frac{1}{2}r_{2*}$$
$$r_{2*} = 1 + \frac{1}{2}\left(1 + \frac{1}{2}r_{2*}\right) = \frac{3}{2} + \frac{1}{4}r_{2*}.$$

So $\frac{3}{4}r_{2*} = \frac{3}{2}$, hence $r_{2*} = 2$.

5. You roll a die until you get the sequence 1,2,1,2. What is the transition matrix for the Markov system describing this game?

   SOLUTION:

$$\begin{bmatrix} \frac{5}{6} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{4}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{5}{6} & 0 & 0 & \frac{1}{6} & 0 \\ \frac{4}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$