Math 367 Exam 2 Solutions Fall '99

1. Roll a pair of dice repeatedly until you get doubles five times. What is the expected number of rolls?

   Solution: We seek the expected number of Bernoulli trials needed for five successes, where the probability, p, of success on a single trial is $\frac{1}{6}$ (the probability of getting doubles on one roll of the dice). The expected value is $\frac{5}{p} = \frac{5}{1/6} = 30$.

2. Pick a letter from

   MISSISSIPPI

   Put it back. Then keep drawing with replacement until you get the same letter again. What is the expected number of draws you make (including the first one)?

   Solution: MISSISSIPPI has 11 letters, so the probability of drawing an M is $\frac{1}{11}$, the probability of drawing an I is $\frac{4}{11}$, the probability of drawing an S is $\frac{4}{11}$, and the probability of drawing a P is $\frac{2}{11}$.

   If the probability of drawing a particular letter is p, the expected number of additional draws needed to draw another one is $\frac{1}{p}$. So adding back the first draw, the expected total number of draws made is $\frac{1}{p} + 1$. Thus,

$$
\begin{aligned}
\mathrm{E}(X) &= \mathrm{E}(X\,|\,\mathrm{M})\mathrm{P}(\mathrm{M}) + \mathrm{E}(X\,|\,\mathrm{I})\mathrm{P}(\mathrm{I}) + \mathrm{E}(X\,|\,\mathrm{S})\mathrm{P}(\mathrm{S}) + \mathrm{E}(X\,|\,\mathrm{P})\mathrm{P}(\mathrm{P}) \\
&= \left(\frac{1}{1/11} + 1\right)\frac{1}{11} + \left(\frac{1}{4/11} + 1\right)\frac{4}{11} + \left(\frac{1}{4/11} + 1\right)\frac{4}{11} \\
&\qquad + \left(\frac{1}{2/11} + 1\right)\frac{2}{11} \\
&= 5.
\end{aligned}
$$

3. Find
   a) the expected value
   b) the variance
   for the number of different denominations (face values) in a 7-card poker hand.

   SOLUTION: Let X be the number of denominations in the hand and let Y be the number of denominations not in the hand. Then $X = 13 - Y$, as there are 13 different denominations.

   Thus, $\mathrm{E}(X) = 13 - \mathrm{E}(Y)$, and $\mathrm{Var}(X) = \mathrm{Var}(Y)$. Also, $Y = Y_1 + \cdots + Y_{13}$, where

$$Y_i = \begin{cases} 1 & \text{if the } i\text{-th denomination is not in the hand} \\ 0 & \text{otherwise.} \end{cases}$$

   Then

$$\begin{aligned} \mathrm{E}(Y_i) &= \mathrm{P}(i\text{-th denomination is not in the hand}) \\ &= \frac{\binom{48}{7}}{\binom{52}{7}}, \end{aligned}$$

   as there are 48 cards not of the i-th denomination. Similarly, if $i \neq j$,

$$\begin{aligned} \mathrm{E}(Y_iY_j) &= \mathrm{P}(\text{neither } i\text{-th nor } j\text{-th denominations in the hand}) \\ &= \frac{\binom{44}{7}}{\binom{52}{7}}, \end{aligned}$$

   as there are 44 cards not in either the i-th or the j-th denomination. Now

$$\begin{aligned} \mathrm{E}(X) = 13 - \mathrm{E}(Y) &= 13 - [\mathrm{E}(Y_1) + \cdots + \mathrm{E}(Y_{13})] = 13 - 13 \cdot \frac{\binom{48}{7}}{\binom{52}{7}} \\ &= 13\left(1 - \frac{\binom{48}{7}}{\binom{52}{7}}\right), \quad \text{while} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X) = \mathrm{Var}(Y) &= E(Y^2) - E(Y) \\ &= E(Y_1^2) + \cdots + E(Y_{13}^2) + \sum_{i \neq j} E(Y_iY_j) - (E(Y_1) + \cdots + E(Y_{13}))^2 \\ &= 13 \cdot \frac{\binom{48}{7}}{\binom{52}{7}} + 13 \cdot 12 \cdot \frac{\binom{44}{7}}{\binom{52}{7}} - \left(13 \cdot \frac{\binom{48}{7}}{\binom{52}{7}}\right)^2. \end{aligned}$$

4. Find
   a) the expected value
   b) the variance
   for the number of hearts in a 7-card poker hand.

   SOLUTION: Here, $X = X_1 + \cdots + X_7$, where

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th card is a heart} \\ 0 & \text{otherwise.} \end{cases}$$

   Then $E(X_i) = \mathrm{P}(i\text{-th card is a heart}) = \frac{13}{52} = \frac{1}{4}$, and for $i \neq j$,

$$\begin{aligned} E(X_iX_j) &= \mathrm{P}(i\text{-th and } j\text{-th cards are hearts}) \\ &= \mathrm{P}(i\text{-th is heart} \,|\, j\text{-th is heart}) \cdot \mathrm{P}(j\text{-th is heart}) \\ &= \frac{12}{51} \cdot \frac{13}{52} = \frac{1}{17}. \end{aligned}$$

   So $\mathrm{E}(X) = E(X_1) + \cdots + E(X_7) = 7 \cdot \frac{1}{4}$, and

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - E(X)^2 \\ &= E(X_1^2) + \cdots + E(X_7^2) + \sum_{i \neq j} E(X_iX_j) - (E(X_1) + \cdots + E(X_7))^2 \\ &= 7 \cdot \frac{1}{4} + 7 \cdot 6 \cdot \frac{1}{17} - (7 \cdot \frac{1}{4})^2, \end{aligned}$$

   as $X_i^2 = X_i$ for all i.

5. An urn contains 10 balls each of 5 different colors. Draw 3 times with replacement. Find the
   a) the expected value
   b) the variance
   for the number of different colors drawn.

   SOLUTION: Let X be the number of colors drawn and let Y be the number of colors not drawn. Then $X = 5 - Y$, as there are 5 colors.

   Thus, $\mathrm{E}(X) = 5 - \mathrm{E}(Y)$, and $\mathrm{Var}(X) = \mathrm{Var}(Y)$. Also, $Y = Y_1 + \cdots + Y_5$, where

$$Y_i = \begin{cases} 1 & \text{if the } i\text{-th color is not drawn} \\ 0 & \text{otherwise.} \end{cases}$$

Since we're drawing with replacement, the probability the i-th color is not drawn in three draws is the third power of the probability the i-th color isn't drawn on a single draw:

$$\begin{aligned} \mathrm{E}(Y_i) &= \mathrm{P}(i\text{-th color is not drawn}) \\ &= \left(\frac{40}{50}\right)^3 = \left(\frac{4}{5}\right)^3. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{E}(Y_iY_j) &= \mathrm{P}(\text{neither } i\text{-th nor } j\text{-th color drawn}) \\ &= \left(\frac{30}{50}\right)^3 = \left(\frac{3}{5}\right)^3. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{E}(X) &= 5 - \mathrm{E}(Y) = 5 - [\mathrm{E}(Y_1) + \cdots + \mathrm{E}(Y_5)] = 5 - 5 \cdot \left(\frac{4}{5}\right)^3 \\ &= 5\left(1 - \left(\frac{4}{5}\right)^3\right), \quad \text{while} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{Var}(Y) = E(Y^2) - E(Y) \\ &= E(Y_1^2) + \cdots + E(Y_5^2) + \sum_{i \neq j} E(Y_iY_j) - (E(Y_1) + \cdots + E(Y_5))^2 \\ &= 5 \cdot \left(\frac{4}{5}\right)^3 + 5 \cdot 4 \cdot \left(\frac{3}{5}\right)^3 - \left(5 \cdot \left(\frac{4}{5}\right)^3\right)^2. \end{aligned}$$

6. A city gets a major earthquake about once every 7 years. What is the probability it will have
a) no major earthquakes
b) more than one major earthquake
in 2001?

SOLUTION: The expected number of major earthquakes per year is $\frac{1}{7}$. We use the Poisson distribution with $m = \frac{1}{7}$. Thus, the probability of k major earthquakes in a given year is $\frac{(1/7)^k}{k!} \cdot e^{-\frac{1}{7}}$.

So the probability of no major quakes is $e^{-\frac{1}{7}}$, and the probability of more than one major quake is

$$\begin{aligned} \mathrm{P}(k > 1) &= 1 - [\mathrm{P}(k = 0) + P(k = 1)] \\ &= 1 - [e^{-\frac{1}{7}} + \frac{1}{7} e^{-\frac{1}{7}}] \\ &= 1 - (1 + \frac{1}{7}) e^{-\frac{1}{7}}. \end{aligned}$$