Math 367 Exam 2 Solutions Summer '98

1. A marksman gets bullseyes on 75% of his shots. He decides to shoot until he gets 10 bullseyes. How many times should he expect to shoot?

   SOLUTION: This is the Pascal distribution with $r = 10$ and $p = .75$. The expected value is $\frac{r}{p} = \frac{40}{3}$.

2. Roll 30 dice. What are the
   a) expected value, and
   b) variance
   for the sum of the numbers on the dice?

   SOLUTION: Let X_i be the number of spots on the i-th die. Then we are concerned with the expected value and variance of

   $$X = X_1 + \cdots + X_{30}.$$

   So $E(X) = E(X_1) + \cdots + E(X_{30})$. Also, X_i and X_j are independent for $i \neq j$, so $\text{Var}(X) = \text{Var}(X_1) + \cdots + \text{Var}(X_{30})$.

   Thus, we need to calculate $E(X_i)$ and $\text{Var}(X_i)$. We have

   $$\begin{aligned} E(X_i) &= 1 \cdot P(X_i = 1) + \cdots + 6 \cdot P(X_i = 6) \\ &= 1 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = 3.5 \end{aligned}$$

   Thus,

   $$\begin{aligned} E(X_i^2) &= 1^2 \cdot P(X_i = 1) + \cdots + 6^2 \cdot P(X_i = 6) \\ &= \frac{1}{6} \cdot (1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6} \end{aligned}$$

   so

   $$\text{Var}(X_i) = E(X_i^2) - (E(X_i))^2 = \frac{35}{12}$$

   Thus,
   a) $E(X) = 30 \cdot E(X_1) = 105$
   b) $\text{Var}(X) = 30 \cdot \text{Var}(X_1) = 87.5$

3. What are the
   a) expected value, and
   b) variance
   for the number of aces in a 7 card poker hand?

   SOLUTION: Let
$$X_i = \begin{cases} 1 & \text{if the i-th card is an ace} \\ 0 & \text{otherwise.} \end{cases}$$
   Then the number of aces in the hand is $X = X_1 + \cdots + X_7$. We have $E(X_i) = P(\text{i-th card is an ace}) = \frac{4}{52} = \frac{1}{13}$. Thus, part a) is answered by
$$E(X) = E(X_1) + \cdots + E(X_7) = 7 \cdot \frac{1}{13} = \frac{7}{13}.$$
   Now, for $i \neq j$,
$$\begin{aligned} E(X_i X_j) &= P(\text{the i-th and j-th cards are aces}) \\ &= P(\text{j-th is ace} \mid \text{i-th is ace}) \cdot P(\text{i-th is ace}) \\ &= \frac{3}{51}\frac{4}{52} = \frac{1}{17 \cdot 13}. \end{aligned}$$
   Because the X_i are characteristic functions, we have
$$\begin{aligned} \text{Var}(X) &= \sum_{i \neq j} E(X_i X_j) + E(X) - (E(X))^2 \\ &= 7 \cdot 6 \cdot \frac{1}{17 \cdot 13} + \frac{7}{13} - (\frac{7}{13})^2 \approx .4385659590 \end{aligned}$$

4. What are the
   a) expected value, and
   b) variance
   for the number of suits in a 7 card poker hand?

   SOLUTION: First order the suits, and then set

$$X_i = \begin{cases} 1 & \text{if the i-th suit is present in the hand} \\ 0 & \text{otherwise.} \end{cases}$$

   Then the number of suits in the hand is $X = X_1 + \cdots + X_4$. Now

$$\begin{aligned} E(X_i) &= 1 - P(\text{the i-th suit is not present in the hand}) \\ &= 1 - \frac{\binom{39}{7}}{\binom{52}{7}} \approx .8850320470 \end{aligned}$$

   Thus,

$$E(X) = 4 \cdot E(X_1) \approx 3.540128188$$

   We next calculate $E(X_i X_j)$ for $i \neq j$. Let C be the event that the i-th suit is not represented in the hand, and let D be the event that the j-th suit is not represented in the hand. Then

$$\begin{aligned} E(X_i X_j) &= 1 - P(C \cup D) \\ &= 1 - [P(C) + P(D) - P(C \cap D)] \\ &= 1 - \left[\frac{\binom{39}{7}}{\binom{52}{7}} + \frac{\binom{39}{7}}{\binom{52}{7}} - \frac{\binom{26}{7}}{\binom{52}{7}} \right] \\ &\approx .7749809545 \end{aligned}$$

   Finally, because the X_i are characteristic functions, we have

$$\begin{aligned} \operatorname{Var}(X) &= \sum_{i \neq j} E(X_i X_j) + E(X) - (E(X))^2 \\ &\approx 4 \cdot 3 \cdot .7749809545 + 3.540128188 - (3.540128188)^2 \\ &\approx .30739205 \end{aligned}$$

5. Flip 3 coins. Then roll as many dice as you got heads. What is the expected value of the sum of the numbers on the dice?

   SOLUTION: Let X be the sum of the numbers on the dice. Let Y be the number produced by the roll of a single die. As shown above, $E(Y) = 3.5$. Also as shown above, if k dice are rolled, then the expected value for the sum of the numbers on the dice is $k \cdot E(Y) = k \cdot 3.5$.

The probability of k heads in three tosses is $\binom{3}{k}(\frac{1}{2})^k(\frac{1}{2})^{3-k} = \binom{3}{k} \cdot \frac{1}{8}$, from the binomial distribution. Thus,

$$\begin{aligned}E(X) &= E(X \mid 0H)P(0H) + \cdots + E(X \mid 3H)P(3H)\\ &= 0 \cdot 3.5 \cdot \frac{1}{8} + 1 \cdot 3.5 \cdot \frac{3}{8} + 2 \cdot 3.5 \cdot \frac{3}{8} + 3 \cdot 3.5 \cdot \frac{1}{8}\\ &= 5.25\end{aligned}$$

6. A 500 page book contains 1200 misprints. Use the Poisson distribution to answer the following:
   a) What is the probability there are no misprints on page 29?
   b) What is the probability there are three or more misprints on page 29?

   Solution: The expected value of misprints per page is $\frac{1200}{500} = 2.4$. Thus, in the Poisson distribution,

$$P(X = k) = e^{-2.4} \cdot \frac{(2.4)^k}{k!}$$

   Thus, the solution to a) is $e^{-2.4} \approx .09071$. For b), we have

$$\begin{aligned}P(X \geq 3) &= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]\\ &= 1 - e^{-2.4}\left(1 + 2.4 + \frac{(2.4)^2}{2}\right)\\ &\approx .43029\end{aligned}$$