1. A marksman gets bullseyes on 75% of his shots. He decides to shoot until he gets 10 bullseyes. How many times should he expect to shoot?

**Solution:** This is the Pascal distribution with $r = 10$ and $p = \frac{4}{5}$. The expected value is $\frac{r}{p} = \frac{40}{3}$.

2. Roll 30 dice. What are the a) expected value, and b) variance for the sum of the numbers on the dice?

**Solution:** Let $X_i$ be the number of spots on the $i$-th die. Then we are concerned with the expected value and variance of $X = X_1 + \cdots + X_{30}$.

So $E(X) = E(X_1) + \cdots + E(X_{30})$. Also, $X_i$ and $X_j$ are independent for $i \neq j$, so $\text{Var}(X) = \text{Var}(X_1) + \cdots + \text{Var}(X_{30})$.

Thus, we need to calculate $E(X_i)$ and $\text{Var}(X_i)$. We have

$$
E(X_i) = 1 \cdot P(X_i = 1) + \cdots + 6 \cdot P(X_i = 6) = 1 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = 3.5$$

Thus,$$
E(X_i^2) = 1^2 \cdot P(X_i = 1) + \cdots + 6^2 \cdot P(X_i = 6)
= \frac{1}{6} \cdot (1 + 4 + 9 + 16 + 25 + 36)
= \frac{91}{6}
$$

so

$$
\text{Var}(X_i) = E(X_i^2) - (E(X_i))^2 = \frac{35}{12}
$$

Thus,

a) $E(X) = 30 \cdot E(X_1) = 105$

b) $\text{Var}(X) = 30 \cdot \text{Var}(X_1) = 87.5$
3. What are the
   a) expected value, and
   b) variance
   for the number of aces in a 7 card poker hand?

**Solution:** Let

\[ X_i = \begin{cases} 
1 & \text{if the } i\text{-th card is an ace} \\
0 & \text{otherwise.} 
\end{cases} \]

Then the number of aces in the hand is \( X = X_1 + \cdots + X_7 \). We have \( E(X_i) = P(\text{i-th card is an ace}) = \frac{4}{52} = \frac{1}{13} \). Thus, part a) is answered by

\[ E(X) = E(X_1) + \cdots + E(X_7) = 7 \cdot \frac{1}{13} = \frac{7}{13}. \]

Now, for \( i \neq j \),

\[ E(X_iX_j) = P(\text{the } i\text{-th and } j\text{-th cards are aces}) \]

\[ = P(\text{j-th is ace} | \text{i-th is ace}) \cdot P(\text{i-th is ace}) \]

\[ = \frac{3 \cdot 4}{51 \cdot 52} = \frac{1}{17 \cdot 13}. \]

Because the \( X_i \) are characteristic functions, we have

\[ \text{Var}(X) = \sum_{i \neq j} E(X_iX_j) + E(X) - (E(X))^2 \]

\[ = 7 \cdot 6 \cdot \frac{1}{17 \cdot 13} + 7 \cdot \frac{1}{13} - \left( \frac{7}{13} \right)^2 \approx .4385659590 \]
4. What are the
   a) expected value, and
   b) variance
for the number of suits in a 7 card poker hand?

SOLUTION: First order the suits, and then set
   \[ X_i = \begin{cases} 
   1 & \text{if the } i\text{-th suit is present in the hand} \\
   0 & \text{otherwise.} 
\end{cases} \]
Then the number of suits in the hand is \( X = X_1 + \cdots + X_4 \). Now
   \[ E(X_i) = 1 - P(\text{the } i\text{-th suit is not present in the hand}) \]
   \[ = 1 - \left( \frac{39}{52} \right) = 1 - \frac{3}{4} = \frac{1}{4} \]
We next calculate \( E(X_i X_j) \) for \( i \neq j \). Let \( C \) be the event that the \( i \)-th suit is not represented in the hand, and let \( D \) be the event that the \( j \)-th suit is not represented in the hand. Then
   \[ E(X_i X_j) = 1 - P(C \cup D) \]
   \[ = 1 - \left[ P(C) + P(D) - P(C \cap D) \right] \]
   \[ = 1 - \left[ \frac{39}{52} + \frac{39}{52} - \frac{26}{52} \right] \]
   \[ = 1 - \left( \frac{39}{52} + \frac{39}{52} - \frac{26}{52} \right) \approx .7749809545 \]
Finally, because the \( X_i \) are characteristic functions, we have
   \[ \text{Var}(X) = \sum_{i \neq j} E(X_i X_j) + E(X) - (E(X))^2 \]
   \[ \approx 4 \cdot 3 \cdot .7749809545 + 3.540128188 - (3.540128188)^2 \]
   \[ \approx .30739205 \]

5. Flip 3 coins. Then roll as many dice as you got heads. What is the expected value of the sum of the numbers on the dice?

SOLUTION: Let \( X \) be the sum of the numbers on the dice. Let \( Y \) be the number produced by the roll of a single die. As shown above, \( E(Y) = 3.5 \). Also as shown above, if \( k \) dice are rolled, then the expected value for the sum of the numbers on the dice is \( k \cdot E(Y) = k \cdot 3.5 \).
The probability of $k$ heads in three tosses is $\binom{3}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{3-k} = \binom{3}{k} \cdot \frac{1}{8}$, from the binomial distribution. Thus,

$$E(X) = E(X \mid 0H)P(0H) + \cdots + E(X \mid 3H)P(3H)$$

$$= 0 \cdot 3.5 \cdot \frac{1}{8} + 1 \cdot 3.5 \cdot \frac{3}{8} + 2 \cdot 3.5 \cdot \frac{3}{8} + 3 \cdot 3.5 \cdot \frac{1}{8}$$

$$= 5.25$$

6. A 500 page book contains 1200 misprints. Use the Poisson distribution to answer the following:

a) What is the probability there are no misprints on page 29?
b) What is the probability there are three or more misprints on page 29?

**Solution:** The expected value of misprints per page is $\frac{1200}{500} = 2.4$.

Thus, in the Poisson distribution,

$$P(X = k) = e^{-2.4} \cdot \frac{(2.4)^k}{k!}$$

Thus, the solution to a) is $e^{-2.4} \approx .09071$. For b), we have

$$P(X \geq 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - e^{-2.4} \left(1 + 2.4 + \frac{(2.4)^2}{2}\right)$$

$$\approx .43029$$