Math 367 **Exam 2** **Fall '02**

1. Flip three coins. Then draw with replacement from a word (specified below) until you get **one more vowel than the number of heads you got**. The word to be used also depends on the number of heads thrown:

Number of heads thrown	Word to use
0H	LION
1H	HIPPO
2H	OSTRICH
3H	ELEPHANT

What is the expected number of letters drawn?

SOLUTION: We need to calculate the conditional expected values of X. For example, the expected value given that two heads were thrown is the expected number of draws needed to get three vowels when drawing with replacement from OSTRICH, i.e., $\frac{3}{p}$, where p is the probability of drawing a vowel with a single draw from OSTRICH. We obtain:

$$E(X \mid 2H) = \frac{3}{2/7} = \frac{21}{2}.$$

The complete calculation is

$$\begin{aligned} E(X) &= E(X \mid 0H)P(0H) + E(X \mid 1H)P(1H) \\ &\quad + E(X \mid 2H)P(2H) + E(X \mid 3H)P(3H) \\ &= \frac{1}{2/4} \cdot \frac{1}{8} + \frac{2}{2/5} \cdot \frac{3}{8} + \frac{3}{2/7} \cdot \frac{3}{8} + \frac{4}{3/8} \cdot \frac{1}{8} \\ &= \frac{2}{8} + \frac{15}{8} + \frac{63}{16} + \frac{32}{24} = \frac{355}{48}. \end{aligned}$$

2. Find
   a) the expected value
   b) the variance
   for the number of suits in a 6-card hand.

SOLUTION: Here, $X = 4 - Y$, where Y is the number of suits not in the hand, and $Y = Y_1 + Y_2 + Y_3 + Y_4$, where

$$Y_i = \begin{cases} 1 & \text{if the } i\text{-th suit is not in the hand} \\ 0 & \text{otherwise.} \end{cases}$$

Because there are 39 cards not of the i-th suit and we're picking 6,

$$E(Y_i) = P(i\text{-th suit not in hand}) = \frac{\binom{39}{6}}{\binom{52}{6}}.$$

Also, if $i \neq j$, there are 26 cards in neither the i-th nor the j-th suit, so

$$E(Y_iY_j) = P(\text{neither } i\text{-th nor } j\text{-th suits in hand}) = \frac{\binom{26}{6}}{\binom{52}{6}}.$$

We have

$$E(Y) = E(Y_1) + E(Y_2) + E(Y_3) + E(Y_4) = 4\frac{\binom{39}{6}}{\binom{52}{6}}, \quad \text{so}$$

$$E(X) = 4 - E(Y) = 4\left[1 - \frac{\binom{39}{6}}{\binom{52}{6}}\right].$$

For the variance,

$$\begin{aligned} \operatorname{Var}(X) = \operatorname{Var}(Y) &= E(Y) - E(Y)^2 + 2\sum_{i<j} E(Y_iY_j) \\ &= \frac{\binom{39}{6}}{\binom{52}{6}} - \left(\frac{\binom{39}{6}}{\binom{52}{6}}\right)^2 + 2\frac{\binom{26}{6}}{\binom{52}{6}}\binom{4}{2}, \end{aligned}$$

where the last term is the number of ways to choose two suits out of four.

3. Find
   a) the expected value
   b) the variance
   for the number of aces in a 6-card hand.

   SOLUTION: Here, $X = X_1 + \cdots + X_6$, where

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th card is an ace} \\ 0 & \text{otherwise.} \end{cases}$$

So $E(X_i) = P(i\text{-th card is an ace}) = \frac{4}{52} = \frac{1}{13}$, and if $i \neq j$, then

$$\begin{aligned} E(X_i X_j) &= P(i\text{-th and } j\text{-th cards aces}) \\ &= P(j\text{-th is ace} \mid i\text{-th is ace}) \cdot P(i\text{-th is ace}) \\ &= \frac{3}{51} \cdot \frac{4}{52} = \frac{1}{17} \cdot \frac{1}{13}. \end{aligned}$$

Thus,

$$\begin{aligned} E(X) &= E(X_1) + \cdots + E(X_6) = 6 \cdot \frac{1}{13} = \frac{6}{13}. \\ \mathrm{Var}(X) &= E(X) - E(X)^2 + 2 \sum_{i<j} E(X_i X_j) \\ &= \frac{6}{13} - \left(\frac{6}{13}\right)^2 + 2 \cdot \frac{1}{17} \cdot \frac{1}{13} \cdot \binom{6}{2}. \end{aligned}$$

In this case, the last term comes from choosing two of the six cards to be aces.

4. An urn contains 9 balls each of 7 different colors. Draw 5 times with replacement. Find the
a) the expected value
b) the variance
for the number of different colors drawn.

SOLUTION: As in problem 2, we write $X = 7 - Y$, where Y is the number of colors not drawn, and $Y = Y_1 + \cdots + Y_7$, where

$$Y_i = \begin{cases} 1 & \text{if the } i\text{-th color is not drawn} \\ 0 & \text{otherwise.} \end{cases}$$

Here, the probability that a single draw is not of the i-th color is $\frac{6}{7}$, and the probability it is neither of the i-th nor j-th color is $\frac{5}{7}$. Since we are drawing with replacement,

$$E(Y_i) = \left(\frac{6}{7}\right)^5, \quad \text{and if } i \neq j,$$
$$E(Y_iY_j) = \left(\frac{5}{7}\right)^5.$$

Here, the exponent 5 comes from the number of draws.

Assembling this, we get

$$E(Y) = E(Y_1) + \cdots + E(Y_7) = 7 \cdot \left(\frac{6}{7}\right)^5, \quad \text{so}$$
$$E(X) = 7 - E(Y) = 7\left[1 - \left(\frac{6}{7}\right)^5\right], \quad \text{and}$$
$$\mathrm{Var}(X) = \mathrm{Var}(Y) = E(Y) - E(Y)^2 + 2\sum_{i<j} E(Y_iY_j)$$
$$= 7 \cdot \left(\frac{6}{7}\right)^5 - \left(7 \cdot \left(\frac{6}{7}\right)^5\right)^2 + 2 \cdot \left(\frac{5}{7}\right)^5 \binom{7}{2}.$$

5. A 500-page book has 325 misprints. What is the probability there are at least two misprints on page 79?

SOLUTION: Here, the expected number of misprints on a given page is $\frac{325}{500} = .65$, so the probability of k misprints on a given page is $\frac{.65^k}{k!}e^{-.65}$ from the Poisson distribution. We have

$$\begin{aligned} P(\geq 2 \text{ misprints}) &= 1 - P(< 2 \text{ misprints}) \\ &= 1 - [P(0 \text{ misprints}) + P(1 \text{ misprint})] \\ &= 1 - [e^{-.65} + .65e^{-.65}] = 1 - 1.65e^{-.65}. \end{aligned}$$