Math 367 Exam 2 Solutions Summer '00

1. Flip three coins. Then draw with replacement from a word (specified below) until you get **one more vowel than the number of heads you got**. The word to be used also depends on the number of heads thrown:

Number of heads thrown	Word to use
0H	ZERO
1H	ONE
2H	TWO
3H	THREE

What is the expected number of letters drawn?

SOLUTION: The expected number of draws needed to get r vowels is $\frac{r}{p}$, where p is the probability of drawing a vowel on a single draw. If X is the total number of draws, then

$$\begin{aligned} \mathrm{E}(X) &= \mathrm{E}(X \mid 0\mathrm{H})\mathrm{P}(0\mathrm{H}) + \mathrm{E}(X \mid 1\mathrm{H})\mathrm{P}(1\mathrm{H}) + \mathrm{E}(X \mid 2\mathrm{H})\mathrm{P}(2\mathrm{H}) \\ &\qquad + \mathrm{E}(X \mid 3\mathrm{H})\mathrm{P}(3\mathrm{H}) \\ &= \frac{1}{1/2} \cdot \frac{1}{8} + \frac{2}{2/3} \cdot \frac{3}{8} + \frac{3}{1/3} \cdot \frac{3}{8} + \frac{4}{2/5} \cdot \frac{1}{8}. \end{aligned}$$

2. Find
a) the expected value
b) the variance
for the number of different suits in a 9-card hand.

SOLUTION: Let X be the number of suits in the hand and let Y be the number of suits not in the hand. Then $X = 4 - Y$, as there are 4 suits in a deck.

Thus, $\mathrm{E}(X) = 4 - \mathrm{E}(Y)$, and $\mathrm{Var}(X) = \mathrm{Var}(Y)$. Also, $Y = Y_1 + \cdots + Y_4$, where

$$Y_i = \begin{cases} 1 & \text{if the } i\text{-th suit is not in the hand} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \mathrm{E}(Y_i) &= \mathrm{P}(i\text{-th suit is not in the hand}) \\ &= \frac{\binom{39}{9}}{\binom{52}{9}}, \end{aligned}$$

as there are 39 cards not of the i-th suit. Similarly, if $i \neq j$,

$$\begin{aligned} \mathrm{E}(Y_iY_j) &= \mathrm{P}(\text{neither } i\text{-th nor } j\text{-th suit in the hand}) \\ &= \frac{\binom{26}{9}}{\binom{52}{9}}, \end{aligned}$$

as there are 26 cards not in either the i-th or the j-th suit. Now

$$\begin{aligned} \mathrm{E}(X) = 4 - \mathrm{E}(Y) &= 4 - [\mathrm{E}(Y_1) + \cdots + \mathrm{E}(Y_4)] = 4 - 4 \cdot \frac{\binom{39}{9}}{\binom{52}{9}} \\ &= 4\left(1 - \frac{\binom{39}{9}}{\binom{52}{9}}\right), \quad \text{while} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X) = \mathrm{Var}(Y) &= E(Y^2) - E(Y) \\ &= E(Y_1^2) + \cdots + E(Y_4^2) + \sum_{i \neq j} E(Y_iY_j) - (E(Y_1) + \cdots + E(Y_4))^2 \\ &= 4 \cdot \frac{\binom{39}{9}}{\binom{52}{9}} + 4 \cdot 3 \cdot \frac{\binom{26}{9}}{\binom{52}{9}} - \left(4 \cdot \frac{\binom{39}{9}}{\binom{52}{9}}\right)^2. \end{aligned}$$

3. Find
   a) the expected value
   b) the variance
   for the number of aces in a 9-card hand.

   SOLUTION: Here, $X = X_1 + \cdots + X_9$, where

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th card is an ace} \\ 0 & \text{otherwise.} \end{cases}$$

Then $E(X_i) = \mathrm{P}(i\text{-th card is an ace}) = \frac{4}{52} = \frac{1}{13}$, and for $i \neq j$,

$$\begin{aligned} E(X_iX_j) &= \mathrm{P}(i\text{-th and } j\text{-th cards are aces}) \\ &= \mathrm{P}(i\text{-th is an ace} \mid j\text{-th is an ace}) \cdot \mathrm{P}(j\text{-th is an ace}) \\ &= \frac{3}{51} \cdot \frac{4}{52} = \frac{1}{17 \cdot 13}. \end{aligned}$$

So $\mathrm{E}(X) = E(X_1) + \cdots + E(X_9) = 9 \cdot \frac{1}{13}$, and

$$\begin{aligned}\mathrm{Var}(X) &= E(X^2) - E(X)^2 \\ &= E(X_1^2) + \cdots + E(X_9^2) + \sum_{i \neq j} E(X_i X_j) - (E(X_1) + \cdots + E(X_9))^2 \\ &= 9 \cdot \frac{1}{13} + 9 \cdot 8 \cdot \frac{1}{17 \cdot 13} - (9 \cdot \frac{1}{13})^2,\end{aligned}$$

as $X_i^2 = X_i$ for all i.

4. An urn contains 8 balls each of 7 different colors. Draw 5 times with replacement. Find the
a) the expected value
b) the variance
for the number of different colors drawn.

Solution: Let X be the number of colors drawn and let Y be the number of colors not drawn. Then $X = 7 - Y$, as there are 7 colors.

Thus, $\mathrm{E}(X) = 7 - \mathrm{E}(Y)$, and $\mathrm{Var}(X) = \mathrm{Var}(Y)$. Also, $Y = Y_1 + \cdots + Y_7$, where

$$Y_i = \begin{cases} 1 & \text{if the } i\text{-th color is not drawn} \\ 0 & \text{otherwise.} \end{cases}$$

Since we're drawing with replacement, the probability the i-th color is not drawn in 5 draws is the 5th power of the probability the i-th color isn't drawn on a single draw:

$$\begin{aligned}\mathrm{E}(Y_i) &= \mathrm{P}(i\text{-th color is not drawn}) \\ &= \left(\frac{48}{56}\right)^5 = \left(\frac{6}{7}\right)^5.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{E}(Y_i Y_j) &= \mathrm{P}(\text{neither } i\text{-th nor } j\text{-th color drawn}) \\ &= \left(\frac{40}{56}\right)^5 = \left(\frac{5}{7}\right)^5.\end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{E}(X) &= 7 - \mathrm{E}(Y) = 7 - [\mathrm{E}(Y_1) + \cdots + \mathrm{E}(Y_7)] = 7 - 7 \cdot \left(\frac{6}{7}\right)^5 \\ &= 7\left(1 - \left(\frac{6}{7}\right)^5\right), \quad \text{while} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X) &= \mathrm{Var}(Y) = E(Y^2) - E(Y) \\ &= E(Y_1^2) + \cdots + E(Y_7^2) + \sum_{i \neq j} E(Y_i Y_j) - (E(Y_1) + \cdots + E(Y_7))^2 \\ &= 7 \cdot \left(\frac{6}{7}\right)^5 + 7 \cdot 6 \cdot \left(\frac{5}{7}\right)^5 - \left(7 \cdot \left(\frac{6}{7}\right)^5\right)^2. \end{aligned}$$

5. A chocolate chip cookie vat contains 12,000 chocolate chips and makes 1,000 cookies. What is the probability a randomly chosen cookie will have at least 3 chips in it?

   Solution: We use the Poisson distribution, where the expected number of chips per cookie is $m = 12$. Thus, the probability there are exactly k chips in a cookie is $e^{-12}\frac{(12)^k}{k!}$. Thus,

$$\begin{aligned} \mathrm{P}(\geq 3 \text{ chips}) &= 1 - \mathrm{P}(< 3 \text{ chips}) \\ &= 1 - \left[e^{-12} + e^{-12} \cdot 12 + e^{-12} \cdot \frac{12^2}{2!}\right]. \end{aligned}$$

6. Suppose that

$$f(z) = \frac{2}{(3-z)}$$

is the generating function for a random variable X. Find

a) $\mathrm{P}(X > 0)$ b) $\mathrm{P}(X < 3)$ c) $\mathrm{E}(X)$ d) $\mathrm{Var}(X)$

SOLUTION: $f(z) = 2(3-z)^{-1}$, so

$$f'(z) = 2(-1)(3-z)^{-2}(-1) = 2(3-z)^{-2}$$
$$f''(z) = 2(-2)(3-z)^{-3}(-1) = 4(3-z)^{-3}.$$

We have

$$\mathrm{P}(X = 0) = f(0) = \frac{2}{3}$$
$$\mathrm{P}(X = 1) = f'(0) = \frac{2}{9}$$
$$\mathrm{P}(X = 2) = \frac{f''(0)}{2!} = \frac{4/27}{2} = \frac{2}{27}.$$

Thus,

$$\mathrm{P}(X > 0) = 1 - \mathrm{P}(X = 0) = 1 - \frac{2}{3} = \frac{1}{3}, \text{ and}$$
$$\mathrm{P}(X < 3) = \mathrm{P}(X = 0) + \mathrm{P}(X = 1) + \mathrm{P}(X = 2) = \frac{2}{3} + \frac{2}{9} + \frac{2}{27}.$$

Also,

$$\mathrm{E}(X) = f'(1) = \frac{1}{2}, \text{ and}$$
$$\mathrm{Var}(X) = f''(1) + f'(1) - (f'(1))^2$$
$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4}.$$