Math 367 Exam 1 Solutions Summer '99

1. How many ways can you arrange the letters in

   CHATTANOOGA TENNESSEE

   so that no two vowels are next to each other?

   Solution: The distribution of letters is as follows:

Consonants		Vowels	
C	1	A	3
H	1	O	2
T	3	E	4
N	3		
G	1		
S	2		
	11		9

   We first choose slots in which to place the vowels. The number of ways to do this is the number of ways to distribute 11 $*$'s and 9 $|$'s so that each adjacent pair of $|$'s has at least one $*$ between them. Since there are 8 spaces between $|$'s, this commits the placement of 8 of the $*$'s. Thus, we seek the number of ways to arrange 9 $|$'s and $11 - 8$ $*$'s in a line. This number is

   $$\binom{9+3}{3} = \binom{12}{3}.$$

   We multiply this by the number of ways to arrange the vowels in their slots times the number of ways to arrange the consonants in their slots. The final answer is

   $$\binom{12}{3} \cdot \frac{9!}{3!\,2!\,4!} \cdot \frac{11!}{3!\,3!\,2!}.$$

2. Roll four dice. What is the probability that a 3 is the highest number shown?

   SOLUTION: Let A be the event that each of the numbers rolled is less than or equal to 3, and let B be the event that each of the numbers rolled is less than or equal to 2. Then the event we want is $A - B$. Since $B \subset A$, we have

$$\begin{aligned} P(A-B) &= P(A) - P(B) \\ &= \left(\frac{3}{6}\right)^4 - \left(\frac{2}{6}\right)^4 \\ &= \left(\frac{1}{2}\right)^4 - \left(\frac{1}{3}\right)^4. \end{aligned}$$

3. A card game has 10-card hands. What is the probability you get four of one denomination and at least three of another?

   SOLUTION: The possible distributions of cards by denomination are $4,4,2,\quad 4,4,1,1,\quad 4,3,3,\quad 4,3,2,1,\quad 4,3,1,1,1$. When picking denominations, we have to remember to divide by the number of ways we could reorder our choices. Thus, the number of ways to pick a hand with distribution $4,4,1,1$ is: Choose two denominations for the 4's and choose four of each, and then choose two denominations for the 1's and pick one of each. We can do this in

$$\binom{13}{2}\binom{4}{4}^2\binom{11}{2}\binom{4}{1}^2$$

   ways. ($\binom{13}{2}$ gives the ways to choose the first two denominations, and $\binom{11}{2}$ gives the ways to choose the second two denominations.)

   Thus, the general solution to the problem is

$$\frac{\binom{13}{2}\binom{4}{4}^2\binom{11}{1}\binom{4}{2} + \binom{13}{2}\binom{4}{4}^2\binom{11}{2}\binom{4}{1}^2 + \binom{13}{1}\binom{4}{4}\binom{12}{2}\binom{4}{3}^2 + x + y}{\binom{52}{10}},$$

   where

$$x = \binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{3}\binom{11}{1}\binom{4}{2}\binom{10}{1}\binom{4}{1} \quad \text{and}$$

$$y = \binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{3}\binom{11}{3}\binom{4}{1}^3.$$

4. A pile of cards consists of seven hearts and five diamonds. You draw 5 at random.
   a) What is the probability you get at least 3 hearts?

   SOLUTION:

$$\frac{\binom{7}{3}\binom{5}{2} + \binom{7}{4}\binom{5}{1} + \binom{7}{5}}{\binom{12}{5}}$$

   b) What is the probability you get at least 3 hearts if you also have at least 1 diamond?

   SOLUTION:

$$\begin{aligned} P(\geq 3H \mid \geq 1D) &= \frac{P(\geq 3H \text{ and } \geq 1D)}{P(\geq 1D)} \\ &= \frac{[\binom{7}{3}\binom{5}{2} + \binom{7}{4}\binom{5}{1}]/\binom{12}{5}}{[\binom{7}{0}\binom{5}{5} + \binom{7}{1}\binom{5}{4} + \binom{7}{2}\binom{5}{3} + \binom{7}{3}\binom{5}{2} + \binom{7}{4}\binom{5}{1}]/\binom{12}{5}} \\ &= \frac{\binom{7}{3}\binom{5}{2} + \binom{7}{4}\binom{5}{1}}{\binom{7}{0}\binom{5}{5} + \binom{7}{1}\binom{5}{4} + \binom{7}{2}\binom{5}{3} + \binom{7}{3}\binom{5}{2} + \binom{7}{4}\binom{5}{1}}. \end{aligned}$$

5. A marksman gets bullseyes on 85% of his shots. Suppose he shoots until he gets 15 bullseyes. What is the probability it takes him at most 18 shots?

   SOLUTION: We use the fact that the probability the rth success occurs on the kth trial is $\binom{k-1}{r-1}p^r q^{k-r}$, where p is the probability of success on a given trial and $q = 1 - p$. Here, $r = 8$ and $p = .85$, so

$$\begin{aligned} P(15 \text{ hits in} \leq 18) &= P(15\text{th on }15\text{th}) + P(15\text{th on }16\text{th}) \\ &\quad + P(15\text{th on }17\text{th}) + P(15\text{th on }18\text{th}) \\ &= \binom{14}{14}(.85)^{15} + \binom{15}{14}(.85)^{15}(.15) \\ &\quad + \binom{16}{14}(.85)^{15}(.15)^2 + \binom{17}{14}(.85)^{15}(.15)^3. \end{aligned}$$

6. Urn A has 9 red balls and 1 green one. Urn B has 3 red balls and 7 green ones.

   Flip **four** coins. If you get **two** heads, draw from Urn A. Otherwise, draw from Urn B.

   Then draw a second ball from the same urn without replacement.

   SOLUTION: We describe the tree diagram without graphing it.

   The probability of 2 heads in the coin toss is $\binom{4}{2}(\frac{1}{2})^2(\frac{1}{2})^2 = 6 \cdot \frac{1}{16}$, so there is a $\frac{6}{16} = \frac{3}{8}$ chance of drawing from Urn A and a $\frac{5}{8}$ chance of drawing from Urn B.

   If you draw from Urn A, the probability the first ball is red is .9. If you draw from Urn B, the probability the first ball is red is .3.

   a) If the first ball is red, what is the probability it came from Urn A?

      SOLUTION: The probability the first ball is red is

$$\begin{aligned} P(\text{1st R}) &= P(\text{1st R} \mid \text{Urn A})P(\text{Urn A}) + P(\text{1st R} \mid \text{Urn B})P(\text{Urn B}) \\ &= .9 \cdot \frac{3}{8} + .3 \cdot \frac{5}{8} = \frac{42}{80}. \end{aligned}$$

      Now

$$\begin{aligned} P(\text{Urn A} \mid \text{1st R}) &= \frac{P(\text{Urn A} \cap \text{1stR})}{P(\text{1st R})} \\ &= \frac{P(\text{1st R} \mid \text{Urn A})P(\text{Urn A})}{P(\text{1st R})} \\ &= \frac{.9 \cdot \frac{3}{8}}{.9 \cdot \frac{3}{8} + .3 \cdot \frac{5}{8}} \\ &= \frac{27/80}{42/80} = \frac{27}{42} = \frac{9}{14}. \end{aligned}$$

b) If the first ball is red, what is the probability the second ball is red?

SOLUTION: We now make a conditional tree diagram under the condition the first ball is red. From Part a), we see

$$P(\text{Urn A} \,|\, \text{1st R}) = \frac{9}{14}, \quad \text{and hence}$$
$$P(\text{Urn B} \,|\, \text{1st R}) = \frac{5}{14}.$$

Thus,

$$\begin{aligned} P(\text{2nd R} \,|\, \text{1st R}) &= P(\text{2nd R} \,|\, \text{Urn A} \cap \text{1st R}) P(\text{Urn A} \,|\, \text{1st R}) \\ &\quad + P(\text{2nd R} \,|\, \text{Urn B} \cap \text{1st R}) P(\text{Urn B} \,|\, \text{1st R}) \\ &= \frac{8}{9} \cdot \frac{9}{14} + \frac{2}{9} \frac{5}{14} \\ &= \frac{41}{63} \end{aligned}$$