Math 367 **Exam 1 Solutions** **Fall '99**

1. How many ways can you arrange the letters in

   GREENVILLE MISSISSIPPI

   so that no two vowels are next to each other?

   Solution: The breakdown in letters is as follows:

G	1	E	3
R	1	I	5
N	1		
V	1		
L	2		
M	1		
S	4		
P	2		
Total	13		8

   The number of ways to pick slots for the vowels is equal to the number of ways to arrange 8 bars and 13 stars so that each pair of adjacent bars has at least one star between them. There are 7 spaces between bars, so this forces the placement of 7 stars.

   Thus, the number of ways to pick slots for the vowels is equal to the number of ways to arrange 8 bars and $(13-7)$ stars in a line with no restrictions on placement. There are $\binom{8+6}{8} = \binom{14}{8}$ ways to do this.

   There are $\frac{8!}{3!5!}$ ways to arrange the vowels in their slots and $\frac{13!}{2!4!2!}$ ways to arrange the consonants in their slots. So the total number of possible arrangements of the letters with no two adjacent vowels is

$$\binom{14}{8} \cdot \frac{8!}{3!5!} \cdot \frac{13!}{2!4!2!}.$$

2. Roll five dice. What is the probability that a 4 is the highest number shown?

   Solution: A 4 is the highest number if
   a) All the numbers are ≤ 4, and
   b) Not all the numbers are ≤ 3.
   Since (all numbers ≤ 3) $\subset$ (all numbers ≤ 4), we have

$$\begin{aligned} \text{P(4 is highest)} &= \text{P(all numbers} \leq 4) - \text{P(all numbers} \leq 3) \\ &= \left(\frac{4}{6}\right)^5 - \left(\frac{3}{6}\right)^5. \end{aligned}$$

3. What is the probability of getting a full house in 7 card poker (i.e., you get three cards in one denomination, at least two cards in at least one other denomination, and no denomination has four cards)?

   SOLUTION: The possible distributions of cards in denominations are (3,3,1), (3,2,2), and (3,2,1,1). Thus, the probability of a full house is

$$\frac{\binom{13}{2}\binom{4}{3}^2\binom{11}{1}\binom{4}{1} + \binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{2}^2 + \binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}\binom{11}{2}\binom{4}{1}^2}{\binom{52}{7}}.$$

4. A pile of cards consists of 8 hearts and 11 diamonds. You draw 6 at random.

   a) What is the probability you get more diamonds than hearts?

   SOLUTION: We seek the probability you get at least 4 diamonds. This is the sum of the probabilities of getting four diamonds, five diamonds, and six diamonds:

$$\mathrm{P}(\geq 4\ \mathrm{D}) = \frac{\binom{11}{4}\binom{8}{2} + \binom{11}{5}\binom{8}{1} + \binom{11}{6}\binom{8}{0}}{\binom{19}{6}}$$

   b) What is the probability you get more diamonds than hearts if you know you have at least 2 hearts?

   SOLUTION:

$$\begin{aligned}\mathrm{P}(\geq 4\ \mathrm{D} \mid \geq 2\ \mathrm{H}) &= \frac{\mathrm{P}([\geq 4\ \mathrm{D}] \text{ and } [\geq 2\ \mathrm{H}])}{\mathrm{P}(\geq 2\ \mathrm{H})} \\ &= \frac{\binom{11}{4}\binom{8}{2}/\binom{19}{6}}{[\binom{11}{0}\binom{8}{6} + \binom{11}{1}\binom{8}{5} + \binom{11}{2}\binom{8}{4} + \binom{11}{3}\binom{8}{3} + \binom{11}{4}\binom{8}{2}]/\binom{19}{6}}\end{aligned}$$

5. A marksman gets bullseyes on 85% of his shots.

   a) What is the probability he gets at least 9 bullseyes in 11 shots?

   SOLUTION: Here, we seek the sum of the probabilities of 9, 10, and 11 bullseyes, respectively, in the 11 shots:

$$\binom{11}{9}(.85)^9(.15)^2 + \binom{11}{10}(.85)^{10}(.15)^1 + \binom{11}{11}(.85)^{11}(.15)^0.$$

   b) Suppose he shoots until he gets 9 bullseyes. What is the probability it takes him at most 11 shots?

   SOLUTION: Here, we seek the sum of the probabilities that the 9-th bullseye occurs on the 9-th, 10-th, and 11-th shot, respectively:

$$\binom{9-1}{9-1}(.85)^9(.15)^0 + \binom{10-1}{9-1}(.85)^9(.15)^1 + \binom{11-1}{9-1}(.85)^9(.15)^2.$$

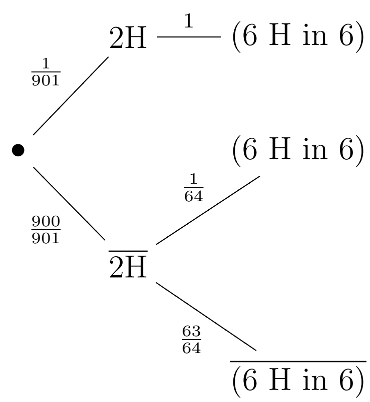
6. A certain country has a careless mint. One out of every 901 coins has two heads. The others are normal.

   Suppose you choose a coin at random and flip it six times. It comes up heads each time.

   a) What is the probability you picked a two-headed coin, given that it comes up heads all six times?

      SOLUTION: Write 2H for the event that you picked the two-headed coin. We get the following tree diagram.

(1)



Now,

$$P(2H \mid (6 \text{ H in } 6)) = \frac{P(2H \text{ and } (6 \text{ H in } 6))}{P(6 \text{ H in } 6)}.$$

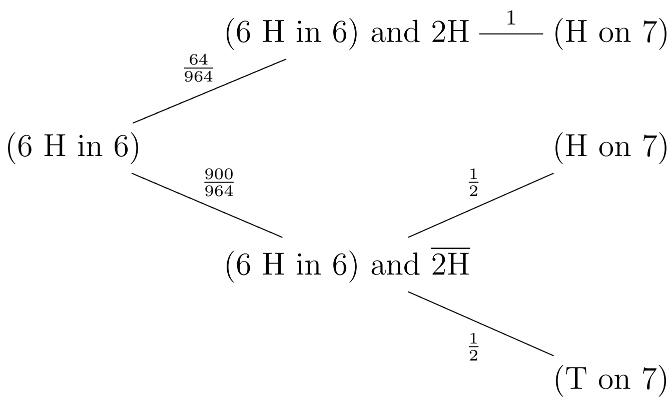
Here, the numerator is the path going through both 2H and (6 H in 6), and the denominator is the sum of all paths ending in (6 H in 6):

$$= \frac{1 \cdot \frac{1}{901}}{1 \cdot \frac{1}{901} + \frac{1}{64} \cdot \frac{900}{901}}$$
$$= \frac{64}{964}.$$

b) What is the probability you get heads on a 7-th toss, given that the first six tosses came up heads?

SOLUTION: We now get a conditional tree diagram:

(2)



We now simply add the paths in this diagram that end with (H on 7):

$$P((H \text{ on } 7) \mid (6 \text{ H in } 6)) = 1 \cdot \frac{64}{964} + \frac{1}{2} \cdot \frac{900}{964} = \frac{514}{964}$$

ALTERNATE SOLUTION:

$$P((H \text{ on } 7) \mid (6 \text{ H in } 6)) = \frac{P((H \text{ on } 7) \text{ and } (6 \text{ H in } 6))}{P(6 \text{ H in } 6)}$$
$$= \frac{P(7 \text{ H in } 7)}{P(6 \text{ H in } 6)}$$

We calculated the denominator in part a). The numerator may be calculated similarly (see below).

$$= \frac{1 \cdot \frac{1}{901} + \frac{1}{128} \cdot \frac{900}{901}}{1 \cdot \frac{1}{901} + \frac{1}{64} \cdot \frac{900}{901}} = \frac{1028}{1928} = \frac{514}{964}.$$

For the calculation of the numerator, we use a tree diagram similar to (1):

(3)