Math 367 Exam 1 Solutions Summer '98

1. How many ways can you arrange the letters in

   WALLA WALLA WASHINGTON

   so that no two vowels are next to each other?

   SOLUTION: There are 7 vowels and 13 consonants. We first pick slots for the vowels. The number of ways to do this is equal to the number of ways to arrange 7 bars and 13 stars in a line so there is at least one star between every pair of bars. To place one star in each such compartment accounts for 6 stars. So the number of ways to pick the slots is equal to the number of ways to arrange 7 bars and 7 stars in a line with no restrictions on position, i.e., $\binom{14}{7}$.

   The vowels consist of 5 A's, an I, and an O, so the number of ways to arrange them in their slots is $7!/5!$. The consonants consist of 3 W's, 4 L's, 1 S, 1 H, 2 N's, 1 G, and 1 T, so the number of ways to arrange them in their slots is $\frac{13!}{3!4!2!}$. Picking slots, and arranging the vowels and consonants in their slots, we get

   $$\binom{14}{7} \cdot \frac{7!}{5!} \cdot \frac{13!}{3!4!2!}$$

   ways to arrange the letters with no two adjacent vowels.

2. Roll three dice. What is the probability that a 5 is the highest number shown?

   SOLUTION: Let A be the event that all three numbers are ≤ 5, and let B be the event that all three numbers are ≤ 4. The event we're interested in is $A-B$. For a general pair of events A and B, $P(A-B) = P(A) - P(A \cap B)$. In this case, $B \subset A$, so $A \cap B = B$. Thus,

   $$\begin{aligned} P(A-B) &= P(A) - P(B) \\ &= \left(\frac{5}{6}\right)^3 - \left(\frac{4}{6}\right)^3 \end{aligned}$$

3. What is the probability you get two pairs in a five card poker hand?

   SOLUTION: Choose two denominations in which to have pairs, choose two cards from each of those denominations, choose a third denomination for the last card, and choose the last card. Then divide by the total number of possible hands. We get

$$\frac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}$$

4. A committee consists of 4 men and 5 women. A four-person subcommittee is chosen at random.

   a) What is the probability there are at least 2 women on the subcommittee?

   SOLUTION:

$$\frac{\binom{5}{2}\binom{4}{2}+\binom{5}{3}\binom{4}{1}+\binom{5}{4}}{\binom{9}{4}}$$

   b) What is the probability there are at least 2 women on the subcommittee if there is at least 1 man on it?

   SOLUTION: Let A be the event that there are at least 2 women on the subcommittee, and let B be the event that there is at least 1 man on it. We wish to calculate $P(A \mid B)$, which by definition is $P(A \cap B)/P(B)$. Now

$$P(A \cap B) = \frac{\binom{5}{2}\binom{4}{2}+\binom{5}{3}\binom{4}{1}}{\binom{9}{4}},$$

   and

$$P(B) = \frac{\binom{4}{4}+\binom{5}{1}\binom{4}{3}+\binom{5}{2}\binom{4}{2}+\binom{5}{3}\binom{4}{1}}{\binom{9}{4}},$$

   so

$$P(A \mid B) = \frac{\binom{5}{2}\binom{4}{2}+\binom{5}{3}\binom{4}{1}}{\binom{4}{4}+\binom{5}{1}\binom{4}{3}+\binom{5}{2}\binom{4}{2}+\binom{5}{3}\binom{4}{1}}.$$

5. A marksman gets bullseyes on 75% of his shots.
   a) Suppose he shoots 12 times. What is the probability he gets at least 10 bullseyes?

   SOLUTION: Let X be the number of bullseyes he hits. Then

$$\begin{aligned} P(X \geq 10) &= P(X = 10) + P(X = 11) + P(X = 12) \\ &= \binom{12}{10}(.75)^{10}(.25)^2 + \binom{12}{11}(.75)^{11}(.25)^1 + (.75)^{12} \\ &\approx .39068 \end{aligned}$$

   b) Suppose he shoots until he gets 10 bullseyes. What is the probability it takes him at most 12 shots?

   SOLUTION: The answer here is the sum of the probabilities that it takes exactly 10, 11, or 12 shots to get 10 bullseyes. From our formula, this is

$$\binom{9}{9}(.75)^{10}(.25)^0 + \binom{10}{9}(.75)^{10}(.25)^1 + \binom{11}{9}(.75)^{10}(.25)^2 \approx .39068$$

6. Urn A has 6 red balls and 4 green ones. Urn B has 3 red balls and 7 green ones.

   Flip **three** coins. If you get **two** heads, draw from Urn A. Otherwise, draw from Urn B.

   Then draw a second ball from the same urn without replacement.

   a) If the first ball is red, what is the probability it came from Urn A?

   SOLUTION: We could draw a tree diagram for this, but will use words instead. The probability of drawing from Urn A is $\binom{3}{2}(\frac{1}{2})^2\frac{1}{2} = \frac{3}{8}$, so the probability of drawing from Urn B is $1 - \frac{3}{8} = \frac{5}{8}$. These would be the initial branches of our tree diagram, and from A and B we would give the conditional probabilities of drawing a red or green ball if our draw comes from that urn. If R is the event that the first ball drawn is red, we have

$$\begin{aligned} P(R) &= P(R \mid A)P(A) + P(R \mid B)P(B) \\ &= \frac{6}{10} \cdot \frac{3}{8} + \frac{3}{10} \cdot \frac{5}{8} = \frac{33}{80}, \text{ so} \\ P(A \mid R) &= \frac{P(A \cap R)}{P(R)} = \frac{P(R \mid A)P(A)}{P(R)} \\ &= \frac{\frac{6}{10} \cdot \frac{3}{8}}{\frac{33}{80}} = \frac{6}{11}. \end{aligned}$$

b) If the first ball is red, what is the probability the second ball is red?

SOLUTION: As above, we have

$$\begin{aligned} P(B \mid R) &= \frac{P(B \cap R)}{P(R)} = \frac{P(R \mid B)P(B)}{P(R)} \\ &= \frac{\frac{3}{10} \cdot \frac{5}{8}}{\frac{33}{80}} = \frac{5}{11}. \end{aligned}$$

We make a new tree, conditional on R having occurred, with branches to A and B, labeled with $P(A \mid R) = \frac{6}{11}$ and $P(B \mid R) = \frac{5}{11}$, respectively. From A and B, we give the conditional probabilities for the second ball being red or green if we're drawing from that urn (the first ball having already been removed). Let R2 be the event that the second ball is red. Then the second tree diagram gives

$$P(R2 \mid R) = \frac{5}{9} \cdot \frac{6}{11} + \frac{2}{9} \cdot \frac{5}{11} = \frac{40}{99}.$$

ALTERNATIVE SOLUTION: $P(R2 \mid R1) = \frac{P(R2 \cap R1)}{P(R1)}$. We've already calculated the denominator. The numerator can be calculated from an absolute tree diagram beginning with the coin toss: If the balls are drawn from Urn A, the probability both are red is $\frac{6}{10} \cdot \frac{5}{9} = \frac{1}{3}$. If the balls are drawn from Urn B, the probability both are red is $\frac{3}{10} \cdot \frac{2}{9} = \frac{1}{15}$. Thus,

$$\begin{aligned} P(R2 \cap R1) &= P(R2 \cap R1 \mid A)P(A) + P(R2 \cap R1 \mid B)P(B) \\ &= \frac{1}{3} \cdot \frac{3}{8} + \frac{1}{15} \cdot \frac{5}{8} = \frac{1}{6}. \end{aligned}$$

Thus,

$$P(R2 \mid R1) = \frac{P(R2 \cap R1)}{P(R1)} = \frac{\frac{1}{6}}{\frac{33}{80}} = \frac{40}{99}.$$