Math 367 Exam 1 Solutions Spring '04

1. How many ways can you arrange the letters in

   WILLIAMSTOWN MASSACHUSETTS

   so that no two vowels are next to each other?

   SOLUTION: We have the following distribution of letters:

W	2	I	2
L	2	A	3
M	2	O	1
S	5	U	1
T	3	E	1
N	1		
C	1		
H	1		
	17		8

   We first find the number of ways to pick 8 slots out of the 25 to put the vowels. These slots must not be adjacent to one another. So the number of ways to pick the slots is the number of ways to arrange 8 |s and 17 ∗s in a row so there is at least one ∗ between every pair of |s. Since there are 7 chambers between |s, we can simply calculate the number of ways to arrange 8 |s and $17-7$ ∗s in a row with no restrictions. This is $\binom{18}{8}$.

   We complete the problem by multiplying this by the number of ways to arrange the vowels in their slots times the number of ways to arrange the consonsnts in theirs. The answer is:

   $$\binom{18}{8}\frac{8!}{2!3!}\frac{17!}{2!2!2!5!3!}.$$

2. a) How many ways can you distribute 20 identical flyers to 5 mailboxes?

   SOLUTION: This is the number of ways to arrange 4 |s (creating 5 chambers) and 20 ∗s in a row: $\binom{24}{4}$.

   b) How many ways can you distribute 20 identical flyers to 5 mailboxes so that each mailbox has at least 2 flyers in it?

   SOLUTION: This time, 10 flyers are committed by the requirement that each box have at least 2. So the whole question is where the remaining 10 flyers to. This is the number of ways to arrange 4 |s and 10 ∗s in a row: $\binom{14}{4}$.

3. A poker-like game has 10-card hands. What is the probability you get three cards in at least two denominations, and no more than three in any denomination?

   SOLUTION: The possible distributions by denomination are: 3,3,3,1; 3,3,2,2; 3,3,2,1,1; 3,3,1,1,1,1. The resulting probability is:

$$\frac{\binom{13}{3}\binom{4}{3}^3\binom{10}{1}\binom{4}{1}+\binom{13}{2}\binom{4}{3}^2\binom{11}{2}\binom{4}{2}^2+\binom{13}{2}\binom{4}{3}^2\binom{11}{1}\binom{4}{2}\binom{10}{2}\binom{4}{1}^2+\binom{13}{2}\binom{4}{3}^2\binom{11}{4}\binom{4}{1}^4}{\binom{52}{10}}$$

4. A committee consists of 8 men and 12 women. You select a 7-member subcommittee at random.
   a) What is the probability you get more women than men?

      SOLUTION: $\frac{\binom{8}{3}\binom{12}{4}+\binom{8}{2}\binom{12}{5}+\binom{8}{1}\binom{12}{6}+\binom{12}{7}}{\binom{20}{7}}$.

   b) What is the probability you get more women than men if you know you have at least 2 women?

      SOLUTION: Here, we want $\frac{\text{P(more W than M and} \geq 2\text{ W)}}{\text{P(}\geq 2\text{ W)}}$. The numerator here, of course, is just P(more W than M). Thus, the desired answer is

$$\frac{\left(\frac{\binom{8}{3}\binom{12}{4}+\binom{8}{2}\binom{12}{5}+\binom{8}{1}\binom{12}{6}+\binom{12}{7}}{\binom{20}{7}}\right)}{\left(\frac{\binom{8}{5}\binom{12}{2}+\binom{8}{4}\binom{12}{3}+\binom{8}{3}\binom{12}{4}+\binom{8}{2}\binom{12}{5}+\binom{8}{1}\binom{12}{6}+\binom{12}{7}}{\binom{20}{7}}\right)}.$$

5. A marksman gets bullseyes on 92% of his shots.
   a) What is the probability he gets at least 15 bullseyes in 17 shots?

      SOLUTION: $\binom{17}{15}(.92)^{15}(.08)^2+\binom{17}{16}(.92)^{16}(.08)^1+(.92)^{17}$.

   b) Suppose he shoots until he gets 15 bullseyes. What is the probability it takes him at most 17 shots?

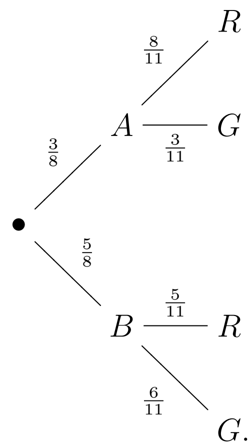
      SOLUTION: $(.92)^{15}+\binom{15}{14}(.92)^{15}(.08)^1+\binom{16}{14}(.92)^{15}(.08)^2$.

6. Urn A has 8 red balls and 3 green balls. Urn B has 5 red balls and 6 green balls.

   Flip a coin 4 times. If you get exactly 2 heads, draw from Urn A. Otherwise, draw from Urn B.

   a) If the ball is red, what is the probability it came from Urn A?

Solution: Solution: The probability of getting 2 heads in 4 tosses is $\binom{4}{2}(\frac{1}{2})^2(\frac{1}{2})^2 = \frac{3}{8}$. We obtain the tree diagram

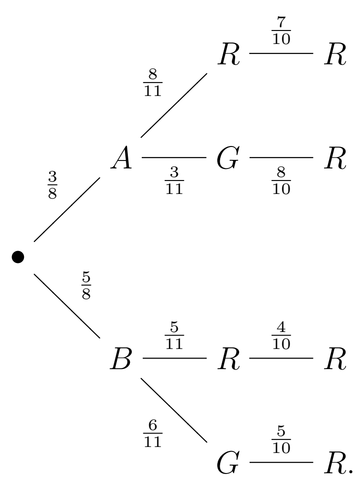


Now,

$$\begin{aligned} P(A \mid R) &= \frac{P(A \cap R)}{P(R)} \\ &= \frac{\frac{3}{8} \cdot \frac{8}{11}}{\frac{3}{8} \cdot \frac{8}{11} + \frac{5}{8} \cdot \frac{5}{11}} = \frac{24}{49}. \end{aligned}$$

b) If the ball is red, what is the probability you get another red ball if you draw without replacement from the same urn?

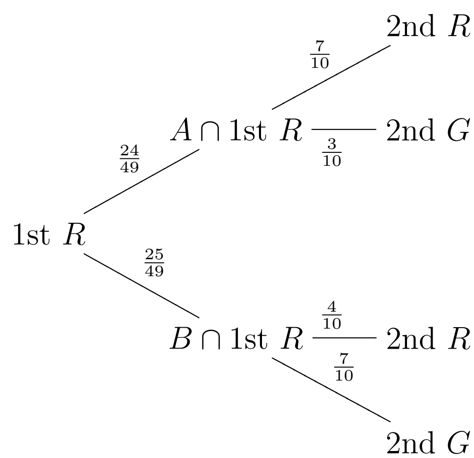
Solution: We first extend the tree diagram, leaving off the outcomes in which the second ball is green:



Now,

$$\begin{aligned} P(2\text{nd } R \mid 1\text{st } R) &= \frac{P(2\text{nd } R \cap 1\text{st } R)}{P(1\text{st } R)} \\ &= \frac{\frac{3}{8}\cdot\frac{8}{11}\cdot\frac{7}{10}+\frac{5}{8}\cdot\frac{5}{11}\cdot\frac{4}{10}}{\frac{3}{8}\cdot\frac{8}{11}+\frac{5}{8}\cdot\frac{5}{11}} = \frac{268}{490} = \frac{134}{245}. \end{aligned}$$

Alternatively, we can make a conditional tree, based on the condition the first ball is red:



This time we add up the paths beginning at 1st R and ending at 2nd R:

$$P(2\text{nd } R \mid 1\text{st } R) = \frac{24}{49}\cdot\frac{7}{10}+\frac{25}{49}\cdot\frac{4}{10} = \frac{134}{245}.$$