Math 367 Exam 1 Solutions Fall '02

1. How many ways can you arrange the letters in

   CHICOPEE MASSACHUSETTS

   so that no two vowels are next to each other?

   Solution: For the consonants, we have 3 C's, 2 H's, 1 P, 1 M, 4 S's and 2 T's for a total of 13 consonants. For vowels, we have 1 I, 1 O, 3 E's, 2 A's and 1 U, for 8 vowels.

   To count the arrangements, we first count the number of ways to pick slots for the vowels. This will be the number of ways to arrange 8 bars and $13 - 7$ stars in a row, or $\binom{14}{8}$.

   The number of ways to arrange the vowels in these slots is $\frac{8!}{3! \cdot 2!}$, where the denominator is the number of ways to rearrange a given arrangement so that only like letters are interchanged. Similarly, the number of ways to arrange the consonants in the remaining slots is $\frac{13!}{3! \cdot 2! \cdot 4! \cdot 2!}$. Thus, the solution to the problem is

   $$\binom{14}{8} \cdot \frac{8!}{3! \cdot 2!} \cdot \frac{13!}{3! \cdot 2! \cdot 4! \cdot 2!}.$$

2. a) How many ways can you distribute 10 identical flyers to 6 mailboxes?

   Solution: This is the number of ways to arrange $6 - 1$ bars and 10 stars in a row, or $\binom{15}{10} = \binom{15}{5}$.

   b) How many ways can you distribute 10 identical flyers to 6 mailboxes so that each mailbox has at least one letter in it?

   Solution: This time, we can throw out six stars, as each compartment must have at least one star in it. Thus, we seek the number of ways to arrange $6 - 1$ bars and 4 stars in a row, or $\binom{9}{4} = \binom{9}{5}$.

3. A poker-like game has 10-card hands. What is the probability you get four cards of one kind and at least three cards of a second kind?

   SOLUTION: The distribution of cards by denominations can be 4,4,2, 4,4,1,1, 4,3,3, 4,3,2,1 or 4,3,1,1,1. The resulting probability is

$$\frac{\binom{13}{2}\binom{4}{4}^2\binom{11}{1}\binom{4}{2}+\binom{13}{2}\binom{4}{4}^2\binom{11}{2}\binom{4}{1}^2+\binom{13}{1}\binom{4}{4}\binom{12}{2}\binom{4}{3}^2}{\binom{52}{10}}$$

$$+\frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{3}\binom{11}{1}\binom{4}{2}\binom{10}{1}\binom{4}{1}+\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{3}\binom{11}{3}\binom{4}{1}^3}{\binom{52}{10}}.$$

4. A committee consists of 7 men and 9 women. You select a 6-member subcommittee at random.
   a) What is the probability you get more women than men?

      SOLUTION: We must have at least four women. The probability is:

$$\frac{\binom{9}{4}\binom{7}{2}+\binom{9}{5}\binom{7}{1}+\binom{9}{6}}{\binom{16}{6}}.$$

   b) What is the probability you get more women than men if you know you have at least 1 man?

      SOLUTION: Here,

$$\mathrm{P}(\text{more women} \mid \geq 1 \text{ man}) = \frac{\mathrm{P}(\text{more women} \cap \geq 1 \text{ man})}{\mathrm{P}(\geq 1 \text{ man})}$$

$$= \frac{\left[\frac{\binom{9}{4}\binom{7}{2}+\binom{9}{5}\binom{7}{1}}{\binom{16}{6}}\right]}{\left[\frac{\binom{7}{6}+\binom{9}{1}\binom{7}{5}+\binom{9}{2}\binom{7}{4}+\binom{9}{3}\binom{7}{3}+\binom{9}{4}\binom{7}{2}+\binom{9}{5}\binom{7}{1}}{\binom{16}{6}}\right]}$$

5. A marksman gets bullseyes on 95% of his shots.
   a) What is the probability he gets at least 12 bullseyes in 14 shots?

      SOLUTION:

$$\mathrm{P}(\geq 12 \text{ B in } 14 \text{ S}) = \mathrm{P}(12 \text{ B in } 14 \text{ S}) + \mathrm{P}(13 \text{ B in } 14 \text{ S}) + \mathrm{P}(14 \text{ B in } 14 \text{ S})$$

$$= \binom{14}{12}(.95)^{12}(.05)^2 + \binom{14}{13}(.95)^{13}(.05) + (.95)^{14}.$$

b) Suppose he shoots until he gets 12 bullseyes. What is the probability it takes him at most 14 shots?

SOLUTION:

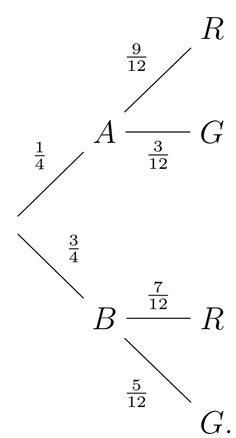
$$\begin{aligned}&\mathrm{P}(\text{12th B on 12th S}) + \mathrm{P}(\text{12th B on 13th S}) + \mathrm{P}(\text{12th B on 14th S})\\ &\quad = \binom{11}{11}(.95)^{12}(.05)^0 + \binom{12}{11}(.95)^{12}(.05)^1 + \binom{13}{11}(.95)^{12}(.05)^2.\end{aligned}$$

6. Urn A has 9 red balls and 3 green balls. Urn B has 7 red balls and 5 green balls.

Flip a coin twice. If you get two heads, draw from Urn A. Otherwise, draw from Urn B.

a) If the ball is red, what is the probability it came from Urn A?

SOLUTION: We have a tree diagram

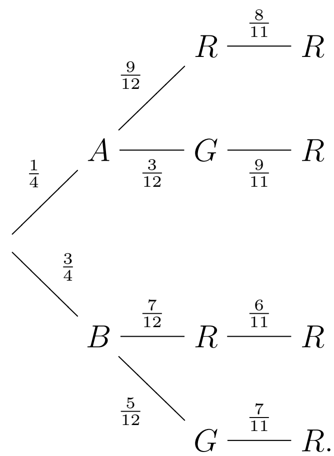


Now,

$$\begin{aligned}\mathrm{P}(A \mid R) &= \frac{\mathrm{P}(A \cap R)}{\mathrm{P}(R)}\\ &= \frac{\frac{1}{4}\cdot\frac{9}{12}}{\frac{1}{4}\cdot\frac{9}{12}+\frac{3}{4}\cdot\frac{7}{12}} = \frac{9}{30} = \frac{3}{10} = .3\ .\end{aligned}$$

b) If the ball is red, what is the probability you get another red ball if you draw without replacement from the same urn?

SOLUTION: We first extend the tree diagram, leaving off the outcomes in which the second ball is green:

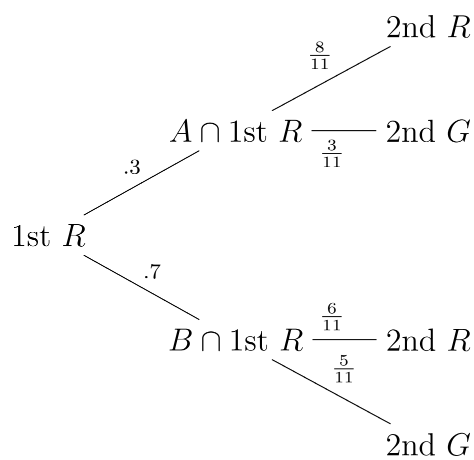


Now,

$$P(\text{2nd } R \mid \text{1st } R) = \frac{P(\text{2nd } R \cap \text{1st } R)}{P(\text{1st } R)}$$

$$= \frac{\frac{1}{4}\cdot\frac{9}{12}\cdot\frac{8}{11} + \frac{3}{4}\cdot\frac{7}{12}\cdot\frac{6}{11}}{\frac{1}{4}\cdot\frac{9}{12} + \frac{3}{4}\cdot\frac{7}{12}} = \frac{3}{5} = .6\ .$$

Alternatively, we can make a conditional tree, based on the condition the first ball is red:



This time we add up the paths beginning at 1st R and ending at 2nd R:

$$P(\text{2nd } R \mid \text{1st } R) = .3\cdot\frac{8}{11} + .7\cdot\frac{6}{11} = \frac{66}{110} = .6\ .$$