Math 367 Exam 1 Solutions Summer '00

1. How many ways can you arrange the letters in

   PHILADELPHIA PENNSYLVANIA

   so that no two vowels are next to each other?

   SOLUTION: The distribution of letters is as follows:

Consonants		Vowels	
P	3	I	3
H	2	A	4
L	3	E	2
D	1	Y	1
N	3		
S	1		
V	1		
	14		10

   We first count the number of ways to pick slots for the vowels. It is equal to the number of ways to arrange 10 $|$'s and 14 $*$'s so that no two $|$'s are next to each other. There are 9 spaces between $|$'s and each must have at least one $*$ in it, so we may throw out 9 of the $*$'s and ask how many ways we can arrange the remaining 5 $*$'s with the 10 $|$'s with no restrictions on the arrangement. There are

$$\frac{(10+5)!}{10! \cdot 5!} = \binom{15}{10} = \binom{15}{5}$$

   ways to do this.

   The answer to our question will be $\binom{15}{5}$ times the number of ways to arrange the vowels in the vowel-slots times the number of ways to arrange the consonants in the consonant-slots. We get

$$\binom{15}{5} \cdot \frac{10!}{3! \cdot 4! \cdot 2!} \cdot \frac{14!}{3! \cdot 2! \cdot 3! \cdot 3!}$$

2. a) How many ways can you distribute 9 different letters to 6 mailboxes?

   SOLUTION: This is the number of ways to chose 9 things from 6 things, where order counts, with replacement: 6^9.

b) How many ways can you distribute 9 identical flyers to 6 mailboxes?

SOLUTION: This is the number of ways to chose 9 things from 6 things, where order does not count, with replacement. Here, we can represent the 6 things by the chambers created by 5 |'s so we seek the number of ways to order 5 |'s and 9 $*$'s. This is just

$$\frac{(5+9)!}{5! \cdot 9!} = \binom{14}{5}$$

3. A poker-like game has 9-card hands. What is the probability you get three cards of at least two kinds, and no more than three of any kind?

SOLUTION: The possible distributions of cards in kinds are 3,3,3, 3,3,2,1 and 3,3,1,1,1. The probability comes to

$$\frac{\binom{13}{3}\binom{4}{3}^3 + \binom{13}{2}\binom{4}{3}^2\binom{11}{1}\binom{4}{3}\binom{10}{1}\binom{4}{1} + \binom{13}{2}\binom{4}{3}^2\binom{11}{3}\binom{4}{1}^3}{\binom{52}{9}}$$

Here, for instance, the number of ways to get a 3,3,1,1,1 distribution is given by (pick two kinds)(pick three of each)(pick three kinds)(pick one of each).

4. A committee consists of 7 Democrats and 8 Republicans. You select a 5-member subcommittee at random.

a) What is the probability you get more Republicans than Democrats?

SOLUTION: You can have 3, 4, or 5 Republicans, so the probability is

$$\frac{\binom{8}{3}\binom{7}{2} + \binom{8}{4}\binom{7}{1} + \binom{8}{5}}{\binom{15}{5}}.$$

b) What is the probability you get more Republicans than Democrats if you know you have at least 1 Republican?

SOLUTION: By definition,

$$P(\geq 3R | \geq 1R) = \frac{P(\geq 3R \cap \geq 1R)}{P(\geq 1R)} = \frac{P(\geq 3R)}{P(\geq 1R)}$$

$$= \frac{[\binom{8}{3}\binom{7}{2} + \binom{8}{4}\binom{7}{1} + \binom{8}{5}]/\binom{15}{5}}{[\binom{8}{1}\binom{7}{4} + \binom{8}{2}\binom{7}{3} + \binom{8}{3}\binom{7}{2} + \binom{8}{4}\binom{7}{1} + \binom{8}{5}]/\binom{15}{5}}$$

5. A marksman gets bullseyes on 80% of his shots.
   a) What is the probability he gets at least 14 bullseyes in 17 shots?

   SOLUTION: We're doing Bernoulli trials with $p = .8$.

$$\begin{aligned}\mathrm{P}(\geq 14 \text{ S in } 17 \text{ T}) &= \mathrm{P}(14 \text{ S in } 17 \text{ T}) + \mathrm{P}(15 \text{ S in } 17 \text{ T}) \\ &\quad + \mathrm{P}(16 \text{ S in } 17 \text{ T}) + \mathrm{P}(17 \text{ S in } 17 \text{ T}) \\ &= \binom{17}{14}(.8)^{14}(.2)^{3} + \binom{17}{15}(.8)^{15}(.2)^{2} \\ &\quad + \binom{17}{16}(.8)^{16}(.2)^{2} + \binom{17}{17}(.8)^{17}(.2)^{0}\end{aligned}$$

   Here S stands for success and T for trial.

   b) Suppose he shoots until he gets 14 bullseyes. What is the probability it takes him at most 17 shots?

   SOLUTION:

$$\begin{aligned}\mathrm{P}(14 \text{ S in } \leq 17 \text{ T}) &= \mathrm{P}(14\text{th S on } 14\text{th T}) + \mathrm{P}(14\text{th S on } 15\text{th T}) \\ &\quad + \mathrm{P}(14\text{th S on } 16\text{th T}) + \mathrm{P}(14\text{th S on } 17\text{th T}) \\ &= \binom{13}{13}(.8)^{14}(.2)^{0} + \binom{14}{13}(.8)^{14}(.2)^{1} \\ &\quad + \binom{15}{13}(.8)^{14}(.2)^{2} + \binom{16}{13}(.8)^{14}(.2)^{3}.\end{aligned}$$

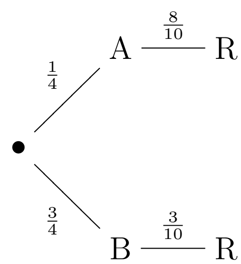
6. Urn A has 8 red balls and 2 green balls. Urn B has 3 red balls and 7 green balls.

   Flip a coin twice. If you get two heads, draw from Urn A. Otherwise, draw from Urn B.

   a) If the ball is red, what is the probability it came from Urn A?

   SOLUTION: The probability we draw from Urn A is the probability of two heads in two tosses: $(\frac{1}{2})^2 = \frac{1}{4}$. We obtain the following tree diagram.

(1)



Thus,

$$\begin{aligned} P(A \mid R) &= \frac{P(A \cap R)}{P(R)} \\ &= \frac{\frac{8}{10} \cdot \frac{1}{4}}{\frac{8}{10} \cdot \frac{1}{4} + \frac{3}{10} \cdot \frac{3}{4}} = \frac{8}{17} \end{aligned}$$

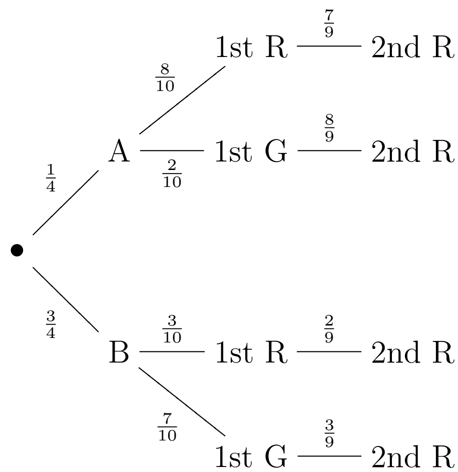
   b) If the ball is red, what is the probability you get another red ball if you draw without replacement from the same urn?

   SOLUTION:

$$P(\text{2nd R} \mid \text{1st R}) = \frac{P(\text{2nd R} \cap \text{1st R})}{P(\text{1st R})}$$

From diagram (1), we have $P(\text{1st R}) = \frac{8}{10} \cdot \frac{1}{4} + \frac{3}{10} \cdot \frac{3}{4}$. To find $P(\text{2nd R} \cap \text{1st R})$, we extend diagram (1) as follows:

(2)



Here, P(2nd R ∩ 1st R) is the sum of the probabilities of the paths that go through both (1st R) and (2nd R). We get

$$\text{P(2nd R} \cap \text{1st R)} = \frac{7}{9} \cdot \frac{8}{10} \cdot \frac{1}{4} + \frac{2}{9} \cdot \frac{3}{10} \cdot \frac{3}{4}.$$

Thus,

$$\begin{aligned} \text{P(2nd R} \mid \text{1st R)} &= \frac{\text{P(2nd R} \cap \text{1st R)}}{\text{P(1st R)}} \\ &= \frac{\frac{7}{9} \cdot \frac{8}{10} \cdot \frac{1}{4} + \frac{2}{9} \cdot \frac{3}{10} \cdot \frac{3}{4}}{\frac{8}{10} \cdot \frac{1}{4} + \frac{3}{10} \cdot \frac{3}{4}} \end{aligned}$$