# Math 362 Final Exam Solutions Fall '09

1. A certain country has a careless mint. One out of every 801 coins comes out with two heads. Pick a coin at random and flip it three times.
   a) What is the probability all three tosses are heads? SOLUTION: $\frac{101}{801}$
   b) What is the conditional probability the coin was two-headed, given that all three tosses were heads? sol: $\frac{1}{101}$.

2. George decides to play roulette, betting on a single number each time, until he wins. Since a roulette wheel has 38 slots, the probability of winning on a given play is $\frac{1}{38}$.
   a) What is the probability he plays at least 5 times? SOLUTION: $q^4 \approx .8988$
   b) How many times should he expect to play? SOLUTION: 38.
   c) What is the variance for the number of plays? SOLUTION: $\frac{q}{p^2} = 37 \cdot 38 = 1406$.

3. Deal 7 cards from a well-shuffled deck.
   a) What is the probability at least five of the cards are spades? SOLUTION: $\frac{\binom{13}{5}\binom{39}{2}+\binom{13}{6}\binom{39}{1}+\binom{13}{7}}{\binom{52}{7}}$
   b) What is the probability at least five of the cards are in the same suit? SOLUTION: $4 \cdot \frac{\binom{13}{5}\binom{39}{2}+\binom{13}{6}\binom{39}{1}+\binom{13}{7}}{\binom{52}{7}}$
   c) How many spades to you expect to get? SOLUTION: $np = 7 \cdot \frac{1}{4} = 1.75$ (hypergeometric).
   d) What is the variance for the number of spades in the hand? SOLUTION: $npq \cdot \frac{N-n}{N-1} = 7 \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{45}{51} = \frac{315}{272} \approx 1.158$.

4. Consider the density function

$$f(x,y) = \begin{cases} \frac{15}{2}xy^2 & \text{for } 0 \le x \le 1,\ -x \le y \le x. \\ 0 & \text{otherwise.} \end{cases}$$

   Compute the following:
   a)

$$f_1(x) = \begin{cases} \int_{-x}^{x} \frac{15}{2}xy^2\,dy = \frac{15}{6}xy^3|_{-x}^{x} = 5x^4 & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

   b) $E(X) = \int_0^1 x f_1(x)\,dx = \int_0^1 5x^5\,dx = \frac{5}{6}$.

c) $V(X)$: Here, $E(X^2) = \int_0^1 x^2 f_1(x)\,dx = \int_0^1 5x^6\,dx = \frac{5}{7}$, so

$$V(X) = E(X^2) - E(X)^2 = \frac{5}{7} - \left(\frac{5}{6}\right)^2 = \frac{5}{252} \approx .0198$$

d) $E(Y|X) = \int_{-x}^{x} y \frac{f(x,y)}{f_1(x)}\,dy = \int_{-x}^{x} \frac{3}{2} \cdot \frac{y^3}{x^3}\,dy = \frac{3}{8} \cdot \frac{y^4}{x^3}\big|_{-x}^{x} = 0.$
$E(Y^2|X) = \int_{-x}^{x} y^2 \frac{f(x,y)}{f_1(x)}\,dy = \int_{-x}^{x} \frac{3}{2} \cdot \frac{y^4}{x^3}\,dy = \frac{3}{10} \cdot \frac{y^5}{x^3}\big|_{-x}^{x} = \frac{3}{5}x^2.$

e) $V(Y|X) = E(Y^2|X) - E(Y|X)^2 = \frac{3}{5}x^2.$

f) $E(Y) = E(E(Y|X)) = \int_0^1 E(Y|X) f_1(x)\,dx = \int_0^1 0\,dx = 0.$

g) $E(Y^2) = E(E(Y^2|X)) = \int_0^1 \frac{3}{5}x^2 \cdot 5x^4\,dx = \frac{3}{7}.$

h) $V(Y) = E(Y^2) - E(Y)^2 = \frac{3}{7}.$

i) $E(XY) = \int_0^1 \int_{-x}^{x} \frac{15}{2}x^2y^3\,dydx = \int_0^2 \frac{15}{8}x^2y^4\big|_{-x}^{x}dx = 0.$

j) $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0.$

k) $\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} = 0.$

l) $E(2X+3Y) = 2E(X) + 3E(Y) = 2\frac{5}{6} + 3 \cdot 0 = \frac{5}{3}.$

m)

$$\begin{aligned} V(2X+3Y) &= 4V(X) + 9V(Y) + 2 \cdot 2 \cdot 3 \cdot \mathrm{Cov}(X,Y) \\ &= 4\frac{5}{252} + 9\frac{3}{7} + 12 \cdot 0 = \frac{248}{63} \approx 3.94\,. \end{aligned}$$

n) If $0 \le y \le 1$, what is $f_2(y)$? SOLUTION: for $0 \le y \le 1$,

$$f_2(y) = \int_y^1 \frac{15}{2}xy^2\,dx = \frac{15}{4}x^2y^2\Big|_y^1 = \frac{15}{4}y^2 - \frac{15}{4}y^4.$$

o) Are $X$ and $Y$ independent? Give a reason. No. $f(x,y) \neq f_1(x)f_2(y)$ for $y > 0$.

p) $P(Y \ge -\frac{1}{4} \mid X \le \frac{1}{2}) = \frac{P(Y \ge -\frac{1}{4}, X \le \frac{1}{2})}{P(X \le \frac{1}{2})}$. Here,

$$\begin{aligned} P\left(Y \ge -\frac{1}{4}, X \le \frac{1}{2}\right) &= \int_0^{\frac{1}{4}} \int_{-x}^{x} \frac{15}{2}xy^2\,dydx + \int_{\frac{1}{4}}^{\frac{1}{2}} \int_{-\frac{1}{4}}^{x} \frac{15}{2}xy^2\,dydx \\ &= \frac{1}{1024} + \frac{77}{4096} = \frac{81}{4096}, \end{aligned}$$

while $P(X \le \frac{1}{2}) = \int_0^{\frac{1}{2}} f_1(x)\,dx = \int_0^{\frac{1}{2}} 5x^4\,dx = \frac{1}{32}.$
So $P(Y \ge -\frac{1}{4} \mid X \le \frac{1}{2}) = \frac{81}{128}.$

q) $P(Y \ge -\frac{1}{4} \mid X = \frac{1}{2}) = \int_{-\frac{1}{4}}^{\frac{1}{2}} \frac{f(\frac{1}{2},y)}{f_1(\frac{1}{2})}\,dy = \int_{-\frac{1}{4}}^{\frac{1}{2}} 12y^2\,dy = \frac{9}{16}$

5. A large jar is filled with coins. 40% of the coins are quarters, 30% are dimes, 20% are nickels and 10% are pennies. Select 10 coins at random.
   a) What is the probability five of the coins are quarters, two are dimes, two are nickels and one is a penny? SOLUTION: $\frac{10!}{5!2!2!1!}(.4)^5(.3)^2(.2)^2(.1)$.
   b) What are the expected value and variance for the amount of money drawn? SOLUTION: let $T$ be the amount of money drawn, $Q$ the number of quarters drawn, $D$ the number of dimes drawn, $N$ the number of nickels drawn and $P$ the number of pennies drawn. then $T = 25Q + 10D + 5N + P$ in cents. We get

$$
\begin{aligned}
E(T) &= 25E(Q) + 10E(D) = 5E(N) + P \\
&= 25 \cdot 10 \cdot (.4) + 10 \cdot 10 \cdot (.3) + 5 \cdot 10 \cdot (.2) + 10 \cdot (.1) = \$1.41\,.
\end{aligned}
$$

$$
\begin{aligned}
V(T) &= 25^2V(Q) + 10^2V(D) + 5^2V(N) + P + 2 \cdot 25 \cdot 10 \cdot \mathrm{Cov}(Q, D) \\
&\quad + 2 \cdot 25 \cdot 5 \cdot \mathrm{Cov}(Q, N) + 2 \cdot 25 \cdot \mathrm{Cov}(Q, P) \\
&\quad + 2 \cdot 10 \cdot 5 \cdot \mathrm{Cov}(D, N) + 2 \cdot 10 \cdot \mathrm{Cov}(D, P) + 2 \cdot 5 \cdot \mathrm{Cov}(N, P) \\
&= \$8.629\,.
\end{aligned}
$$

Here, e.g., $V(Q) = 10(.4)(.6)$ and $\mathrm{Cov}(Q, D) = -10(.4)(.3)$.