

1. Consider the density function

$$f(x, y) = \begin{cases} 10x^2y & \text{for } 0 \leq x \leq 1, 0 \leq y \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the following:

- a)  $f_1(x)$ .  $\int_0^x 10x^2y \, dy = 5x^2y^2|_0^x = 5x^4$ .  $f_1(x) = \begin{cases} 5x^4 & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$
- b)  $E(X)$ .  $\int_0^1 5x^5 \, dx = \frac{5}{6}$ .
- c)  $V(X)$ .  $E(X^2) = \int_0^1 5x^6 \, dx = \frac{5}{7}$ .  $V(X) = \frac{5}{7} - \frac{25}{36} = \frac{5}{252}$ .
- d)  $E(Y)$ .  $\int_0^1 \int_0^x 10x^2y^2 \, dy \, dx = \frac{5}{9}$ .
- e)  $V(Y)$ .  $E(Y^2) = \int_0^1 \int_0^x 10x^2y^3 \, dy \, dx = \frac{5}{14}$ .  $V(Y) = \frac{55}{1134}$ .
- f)  $E(XY)$ .  $\int_0^1 \int_0^x 10x^3y^2 \, dy \, dx = \frac{10}{21}$ .
- g)  $\text{Cov}(X, Y)$ .  $\frac{10}{21} - \frac{25}{45} = \frac{5}{378}$ .
- h)  $\rho(X, Y)$ .  $\frac{5}{378} / \sqrt{\frac{5}{252} \cdot \frac{55}{1134}} = \frac{\sqrt{22}}{11} \approx .4264$
- i)  $E(Y|X)$ .  $\int_0^x \frac{10x^2y^2}{5x^4} \, dy = \frac{2}{3}x$ .
- j)  $V(2X - 3Y)$ .  $4 \cdot \frac{5}{252} + 9 \cdot \frac{55}{1134} - 12 \cdot \frac{5}{378} = \frac{5}{14}$ .
- k) What is the density function for  $U = Y - X$ ? For  $-1 \leq u \leq 0$ ,

$$P(U \leq u) = \int_{-u}^1 \int_0^{x+u} 10x^2y \, dy \, dx = \frac{1}{6}u^5 + \frac{5}{3}u^2 + \frac{5}{2}u + 1.$$

$$f_U(u) = \begin{cases} \frac{5}{6}u^4 + \frac{10}{3}u + \frac{5}{2} & -1 \leq u \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

l) Are  $X$  and  $Y$  independent? No. The active region is not rectangular.

2. Let  $X$  have density function

$$f(x) = \begin{cases} 5x^4 & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the density function for  $U = X^3$ ? For  $0 \leq u \leq 1$ ,

$$P(X^3 \leq u) = P(X \leq u^{\frac{1}{3}}) = \int_0^{u^{\frac{1}{3}}} 5x^4 dx = u^{\frac{5}{3}}.$$

$$f_U(u) = \begin{cases} \frac{5}{3}u^{\frac{2}{3}} & 0 \leq u \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

3. What is the moment generating function of  $5X$ , where  $X$  is a normal distribution with mean 0 and standard deviation 3?

$$m_{5X}(t) = m_X(5t) = e^{\frac{3^2(5t)^2}{2}} = e^{\frac{225t^2}{2}}.$$