Math 362 **Exam 3 Solutions** **Fall '09**

1. Consider the density function

$$f(x,y) = \begin{cases} 8xy & \text{for } 0 \le y \le x \le 1. \\ 0 & \text{otherwise.} \end{cases}$$

Compute the following:

a) $f_1(x)$. Solution:

$$f_1(x) = \begin{cases} \int_0^x 8xy\,dy = 4xy^2|_0^x = 4x^3 & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

b) $f_2(y)$. Solution:

$$f_2(y) = \begin{cases} \int_y^1 8xy\,dx = 4x^2y|_y^1 = 4(y-y^3) & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

c) $E(X)$. Solution:

$$E(X) = \int_0^1 x \cdot 4x^3\,dx = \frac{4}{5}x^5\Big|_0^1 = \frac{4}{5}.$$

d) $V(X)$. Solution:

$$E(X^2) = \int_0^1 x^2 \cdot 4x^3\,dx = \frac{4}{6}x^6\Big|_0^1 = \frac{2}{3}.$$

$$V(X) = E(X^2) - E(X)^2 = \frac{2}{3} - \frac{16}{25} = \frac{2}{75}.$$

e) $E(Y)$. Solution:

$$E(Y) = \int_0^1 y \cdot 4(y-y^3)\,dy = 4\left(\frac{y^3}{3} - \frac{y^5}{5}\right)\Big|_0^1 = \frac{8}{15}.$$

f) $V(Y)$. Solution:

$$E(Y^2) = \int_0^1 y^2 \cdot 4(y-y^3)\,dy = 4\left(\frac{y^4}{4} - \frac{y^6}{6}\right)\Big|_0^1 = \frac{1}{3}.$$

$$V(Y) = E(Y^2) - E(Y)^2 = \frac{1}{3} - \frac{64}{225} = \frac{11}{225}.$$

g) $E(XY)$. Solution:

$$E(XY) = \int_0^1 \int_0^x xy \cdot 8xy\,dydx = \int_0^1 \frac{8}{3}x^2y^3\Big|_0^x\,dx = \frac{8}{18}x^6\Big|_0^1 = \frac{4}{9}.$$

h) Cov(X, Y). SOLUTION:

$$\operatorname{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{4}{9} - \frac{4}{5} \cdot \frac{8}{15} = \frac{4}{225}.$$

i) ρ. SOLUTION:

$$\rho = \frac{\operatorname{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{\frac{4}{225}}{\sqrt{\frac{2}{75} \cdot \frac{11}{225}}} = \frac{4}{\sqrt{66}} \approx .49\,.$$

j) $E(Y|X)$. SOLUTION:

$$E(Y|X) = \int_0^x y \cdot \frac{8xy}{4x^3}\, dy = \frac{2}{3} \cdot \frac{y^3}{x^2}\bigg|_0^x = \frac{2}{3}x.$$

k) $E(Y^2|X)$. SOLUTION:

$$E(Y^2|X) = \int_0^x y^2 \cdot \frac{8xy}{4x^3}\, dy = \frac{1}{2} \cdot \frac{y^4}{x^2}\bigg|_0^x = \frac{1}{2}x^2.$$

l) $V(Y|X)$. SOLUTION:

$$V(Y|X) = E(Y^2|X) - E(Y|X)^2 = x^2\left(\frac{1}{2} - \frac{4}{9}\right) = \frac{1}{18}x^2.$$

m) $E(X+Y)$. SOLUTION:

$$E(X+Y) = E(X) + E(Y) = \frac{4}{5} + \frac{5}{15} = \frac{4}{3}.$$

n) $V(X+Y)$. SOLUTION:

$$V(X+Y) = V(X) + V(Y) + 2\operatorname{Cov}(X,Y) = \frac{2}{75} + \frac{11}{225} + 2\frac{4}{225} = \frac{1}{9}.$$

o) Are X and Y independent? Give a reason. SOLUTION: No. $f(x,y) \neq f_1(x)f_2(y)$.

2. There are four types of hot pepper in a store, of roughly equal size. Someone fills a bag with a selection from the shelf. Each pepper has a different degree of hotness, measured as a multiple of a standard amount. Suppose the percentages in the bag and the hotness ratings of the various types are given as follows:

Pepper type	percentage in the bag	hotness rating
A	45%	4
B	35%	2
C	15%	3
D	5%	1

Pick 6 peppers at random from the bag, and assume there are enough peppers in the bag that the selection may be assumed to have been made with replacement.

a) What is the probability you get two each of types A and B and one each of types C and D? SOLUTION:

$$\frac{6!}{2!2!1!1!}(.45)^2(.35)^2(.15)(.05).$$

b) What are the expected value and variance for the sum of the hotness ratings of the peppers picked? SOLUTION:

$$\begin{aligned}E(H) &= 4E(A) + 2E(B) + 3E(C) + E(D)\\ &= 4\cdot 6\cdot(.45) + 2\cdot 6\cdot(.35) + 3\cdot 6\cdot(.15) + 6\cdot(.05)\\ V(H) &= 16V(A) + 4V(B) + 9V(C) + V(D) + 16\text{Cov}(A,B)\\ &\quad + 24\text{Cov}(A,C) + 8\text{Cov}(A,D) + 12\text{Cov}(B,C) + 4\text{Cov}(B,D)\\ &\quad + 6\text{Cov}(C,D).\end{aligned}$$

Here,

$$\begin{aligned}V(A) &= 6(.45)(.55)\\ V(B) &= 6(.35)(.65)\\ V(C) &= 6(.15)(.85)\\ V(D) &= 6(.05)(.95)\\ \text{Cov}(A,B) &= -6(.45)(.35)\\ \text{Cov}(A,C) &= -6(.45)(.15)\\ \text{Cov}(A,D) &= -6(.45)(.05)\\ \text{Cov}(B,C) &= -6(.35)(.15)\\ \text{Cov}(B,D) &= -6(.35)(.05)\\ \text{Cov}(C,D) &= -6(.15)(.05).\end{aligned}$$