Math 362 **Exam 2** **Spring '11**

1. Consider the density function

$$f(x,y) = \begin{cases} 8xy & \text{for } 0 \le x \le 1,\ 0 \le y \le x \\ 0 & \text{otherwise.} \end{cases}$$

a) What is the marginal density function $f_1(x)$? SOLUTION:

$$f_1(x) = \begin{cases} \int_0^x 8xy\,dy = 4x^3 & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

b) What are $E(X)$ and $V(X)$? SOLUTION:

$$E(X) = \int_0^1 x \cdot 4x^3\,dx = \frac{4}{5},$$
$$E(X^2) = \int_0^1 x^2 \cdot 4x^3\,dx = \frac{4}{6} = \frac{2}{3},$$
$$V(X) = E(X^2) - E(X)^2 = \frac{2}{3} - \frac{16}{25} = \frac{2}{75}.$$

c) What is $P(Y \le \frac{1}{4} \mid X = \frac{1}{2})$? SOLUTION:

$$P\left(Y \le \frac{1}{4} \mid X = \frac{1}{2}\right) = \frac{\int_0^{\frac{1}{4}} f(\frac{1}{2}, y)\,dy}{f_1(\frac{1}{2})}$$
$$= \frac{\int_0^{\frac{1}{4}} 4y\,dy}{(\frac{1}{2})} = \frac{(\frac{1}{8})}{(\frac{1}{2})} = \frac{1}{4}.$$

d) What is $P(X \le \frac{1}{2})$? SOLUTION:

$$P\left(X \le \frac{1}{2}\right) = \int_0^{\frac{1}{2}} f_1(x)\,dx = x^4\Big|_0^{\frac{1}{2}} = \frac{1}{16}.$$

e) What is $P(Y \le \frac{1}{4}, X \le \frac{1}{2})$? SOLUTION: Here, we double integrate over the region ($X \le \frac{1}{2}$ and $Y \le \frac{1}{4}$), which must be broken into two subregions in order to get consistent bounds of integration in the y direction:

$$P\left(Y \le \frac{1}{4}, X \le \frac{1}{2}\right) = \int_0^{\frac{1}{4}} \int_0^x 8xy\,dydx + \int_{\frac{1}{4}}^{\frac{1}{2}} \int_0^{\frac{1}{4}} 8xy\,dydx$$
$$= \int_0^{\frac{1}{4}} 4xy^2\Big|_0^x dx + \int_{\frac{1}{4}}^{\frac{1}{2}} 4xy^2\Big|_0^{\frac{1}{4}} dx$$
$$= \int_0^{\frac{1}{4}} 4x^3\,dx + \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{x}{4}\,dx$$
$$= \frac{1}{256} + \frac{x^2}{8}\Big|_{\frac{1}{4}}^{\frac{1}{2}}$$
$$= \frac{1}{256} + \frac{1}{32} - \frac{1}{128} = \frac{7}{256}.$$

f) What is $P(Y \le \frac{1}{4} \mid X \le \frac{1}{2})$? SOLUTION:

$$P\left(Y \le \frac{1}{4} \mid X \le \frac{1}{2}\right) = \frac{P\left(Y \le \frac{1}{4}, X \le \frac{1}{2}\right)}{P\left(X \le \frac{1}{2}\right)} = \frac{\frac{7}{256}}{\frac{1}{16}} = \frac{7}{16}.$$

g) What is $E(Y)$? SOLUTION:

$$E(Y) = \int_0^1 \int_0^x y \cdot 8xy\,dydx = \int_0^1 \frac{8}{3}xy^3\Big|_0^x dx = \int_0^1 \frac{8}{3}x^4\,dx = \frac{8}{15}.$$

h) What is $E(XY)$? SOLUTION:

$$E(XY) = \int_0^1 \int_0^x xy \cdot 8xy\,dydx = \int_0^1 \frac{8}{3}x^2y^3\Big|_0^x dx = \int_0^1 \frac{8}{3}x^5\,dx = \frac{4}{9}.$$